Nonlinear systems of Differential Equations

1. Introduction to basic concepts

Let's consider the following system

\[
\begin{align*}
\frac{dx}{dt} &= f(x,y) \\
\frac{dy}{dt} &= g(x,y)
\end{align*}
\]

where \( f \) and \( g \) are of class \( C^1 \) on a domain \( D \). The system is called autonomous if \( f \) and \( g \) do not depend explicitly on the variable \( t \). The system can be written in vector notation

\[
X' = F(X) \quad \text{where} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad F(X) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}
\]

Given an initial point \( X(t_0) = X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \), by the existence and uniqueness theorem there is only one solution defined on some interval containing \( t_0 \). The solution \( X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \) is a curve traced by a moving point in the \( xy \) plane or phase plane. This curve is called a trajectory. A representative set of these trajectories is called a phase portrait which is a sketch of the qualitative behavior of the solutions of the system.
We can also construct at chosen points line segments having slope
\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x,y)}{f(x,y)},
\]
and we obtain what is called a direction field or slope field.

**Definition:** Critical point

A point \( C = (c_1, c_2) \), where both \( f \) and \( g \) vanish is called a critical point of the system. At such a point \( \frac{dx}{dt} = 0 = \frac{dy}{dt} \), which means the velocity vector is zero. Therefore the constant valued function \( x(t) = c_1, y(t) = c_2 \) is a solution of the system. It corresponds to an equilibrium state of the solution. It is a point where the motion described by (1) is in a state of rest.

A critical point \( C \) is called an isolated critical point if there is a neighborhood of \( C \) which contains no other critical point.

**Example 1:**
\[
x' = x + y \\
y' = 2x
\]

Sketch the 3 solution curves passing through the following initial points: \( P_1 = (0,0), P_2 = (0,1), P_3 = (1,1) \)
The general solution is
\[ X(t) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} \]
\[ = c_1 K_1 e^{-t} + c_2 K_2 e^{2t} \]

For \( P_1 \) we get
\[ X_1(t) = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \frac{2}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} \]

For \( P_2 \) we get
\[ X_2(t) = \frac{1}{3} \begin{pmatrix} -1 \\ -2 \end{pmatrix} e^{-t} + \frac{1}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} \]

For \( P_3 \) we get
\[ X_3(t) = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix} e^{-t} \]

\[ \begin{align*}
\frac{dx}{dt} &= -x \\
\frac{dy}{dt} &= -ky, \quad k > 0
\end{align*} \]

Example 2
The equations can be solved independently \( \{ x = x_0 e^{-t}, \quad y = y_0 e^{-kt} \} \)

Case when \( k = 1 \) : \( \{ x = x_0 e^{-t}, \quad y = y_0 e^{-t} \} \Rightarrow y = c e^{-t} \)

Case when \( k > 1 \) : \( \text{then } y = c e^{kx} \)
Example 3 \[
\frac{dx}{dt} = 1 \\
\frac{dy}{dt} = -4x \]

Sketch the trajectory which passes through the initial point \( X(0) = X_0 = (0, 0) \).

From what we already know, we can obtain the unique solution \( X(t) = \left( \cos 2t \right) \) which is an ellipse.

Example 4 \[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -2x - 2y \]

Sketch the trajectory which passes through the point \( X_0 = (-1, 1) \).

In this case, the solution turns out to be a spiral \( x = e^{-t} \cos t \), \( y = -e^{-t} (\cos t + \sin t) \).

With these examples in mind, we introduce the following concepts of stability.
Definition: **Stability**
A critical point \( C \) is said to be stable if given \( \epsilon > 0 \) there is \( \delta > 0 \) such that every solution \( X(t) \) of the system (1) which at \( t = 0 \) satisfies \( \| X(0) - C \| < \delta \),

i) exists for all \( t \geq 0 \),

ii) \( \| X(t) - C \| < \epsilon \) for all \( t \geq 0 \).

In other words, all solutions that start "sufficiently close" to the point \( C \) will stay "sufficiently close" to the point \( C \).

Definition: **Asymptotic Stability**
A point \( C \) is called asymptotically stable, if in addition to being stable, every solution \( X(t) \) which at \( t = 0 \) satisfies \( \| X(0) - C \| < \delta \) also satisfies the limiting property \( \lim_{t \to \infty} X(t) = C \).

In other words, every trajectory that starts "sufficiently close" to the point \( C \) must eventually approach \( C \) as \( t \to \infty \).

More examples and discussions which will illustrate these concepts of stability are given in the following sections.
2. **Phase plane. Linear systems**

Let's consider the linear system

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

where \(a, b, c\) and \(d\) are constants, or in vector notation

\[
X' = AX,
\]

where \(X = (x, y)\) and \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\).

If \(\det A \neq 0\), the origin is clearly an isolated critical point.

**Note:** We have already seen that the eigenvalues of \(A\) are exactly the roots of the characteristic equation \(\det (A - \lambda I) = 0\).

**Case 1. Distinct real eigenvalues of the same sign**

We first consider \(\lambda_1 < \lambda_2 < 0\). If \(\mathbf{K}_1\) and \(\mathbf{K}_2\) are the eigenvectors of \(\lambda_1\) and \(\lambda_2\) (resp.) then the general solution is

\[
X = c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_2 t}.
\]

Since \(\lambda_1, \lambda_2 < 0\), \(X(t) \to 0\) as \(t \to \infty\). If \(c_2 = 0\) then \(X(t) \to 0\) along \(\mathbf{K}_1\); otherwise (\(c_2 \neq 0\)) all solutions \(X(t) \to 0\) along \(\mathbf{K}_2\). The origin, in this case, is called a stable node.
On the other hand if $0 < \lambda_1 < \lambda_2$, the trajectories have the same pattern but the direction of motion is away from the origin. In this case, $(0,0)$ is called an unstable node.

\[ (\lambda_1 < \lambda_2 < 0, \text{ stable node}) \]

\[ (0 < \lambda_1 < \lambda_2, \text{ unstable node}) \]

**Case 2. Real eigenvalues of opposite signs**

Suppose $\lambda_2 < 0 < \lambda_1$. If $K_1$ and $K_2$ are the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2$ (resp.), then the general solution is $X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}$. If $c_2 = 0$, then $X = c_1 K_1 e^{\lambda_1 t} \to 0$ as $t \to \infty$ along $K_2$; otherwise, (if $c_1 \neq 0$) all solutions tend to $\infty$ along $K_1$. In this case, the origin is called a saddle point.
Case 3. Equal eigenvalues \( \lambda_1 = \lambda_2 = \lambda \)

a) Case of 2 linearly independent eigenvectors

Suppose \( K_1 \) and \( K_2 \) are the 2 L.I. eigenvectors corresponding to \( \lambda \). The general solution is \( X = (c_1 K_1 + c_2 K_2) e^{\lambda t} \). \( Y \) is independent of \( t \). Every trajectory lies on a line passing through the origin. If \( \lambda < 0 \), all solutions \( X \to 0 \) as \( t \to \infty \). But if \( \lambda > 0 \), then the trajectories are reversed in their direction (away from the origin). The origin is called a proper node.

\( \lambda < 0 \), proper node (stable)

\( \lambda > 0 \), proper node (unstable)
b) Case of only one eigenvector

In this case the general solution is

\[ X = c_1 e^{\lambda t} + c_2 (K_2 + tK_1) e^{\lambda t} \]

where

\[ (A - \lambda I)K_1 = 0 \quad K_1 \neq 0 \]

and

\[ (A - \lambda I)K_2 = K_1 \]

If \( \lambda < 0 \), \( X(t) \) which passes through the point \( c_1 K_1 + c_2 K_2 \) (at \( t = 0 \)) must eventually tend to zero, in the direction of \( K_1 \), as \( t \to \infty \). If \( \lambda > 0 \), the direction of the arrows is reversed (away from the origin). In the case 3(b), (0,0) is called improper node.

\[ \lambda < 0 \]

**Case 4. Case of complex eigenvalues:** \( \lambda = \alpha + i\beta \)

\[ \begin{align*}
  x' &= \alpha x + \beta y \\
  y' &= -\beta x + \alpha y
\end{align*} \]

a) Case of pure imaginary eigenvalue

\[ \begin{align*}
  x' &= \beta y \\
  y' &= -\beta x
\end{align*} \]
The general solution is \[
\begin{align*}
x &= A \cos (\beta t + \phi) \\
y &= A \sin (\beta t + \phi)
\end{align*}
\]

The trajectories are circles of radius $A$ centered at the origin, with period $\frac{2\pi}{\beta}$. If $\beta > 0$, the motion is in the positive direction; but if $\beta < 0$, it is in the negative direction.

In the case $\beta = 0$, the origin is called a **center**.

b) General complex eigenvalue $\lambda = \alpha \pm i \beta$

Using polar coordinates system: $r = \sqrt{x^2 + y^2}$, $\theta = \arctan \frac{y}{x}$, we have

\[
\frac{dr}{dt} = \frac{x}{r} y' + \frac{y}{r} x' = \frac{x}{r} (\alpha x + \beta y) + \frac{y}{r} (-\beta x + \alpha y) = \alpha r
\]

\[
\frac{d\theta}{dt} = \frac{y' - \frac{y}{r} x'}{\frac{x}{r} y} = \frac{x}{x^2 + y^2} (-\beta x + \alpha y) - \frac{y}{x^2 + y^2} (\alpha x + \beta y) = -\beta
\]

\[
\frac{dr}{dt} = -\beta \quad \Rightarrow \quad r = r e^{-\beta t}
\]

Therefore \[
\begin{align*}
x &= r e^{-\beta t} \cos (\alpha t + \phi) \\
y &= r e^{-\beta t} \sin (\alpha t + \phi)
\end{align*}
\]

If $\alpha < 0$, $(x(t), y(t)) \to 0$ as $t \to \infty$. In this case $(0,0)$ is called a **spiral sink**. But if $\alpha > 0$, $\|x(t)\| \to \infty$ as $t \to \infty$. In this case, $(0,0)$ is called a **spiral source**.
Example 1

\[ x' = y \\
\gamma' = x \]

determine the type of critical point at the origin and describe its stability.

The eigenvalues are \( 1 \) and \(-1\).

The general solution is \( X = c_1 (1, 1)e^t + c_2 (1, -1)e^{-t} \)

The origin is a saddle point and is unstable.

The trajectories of the system can also be obtained directly from the system by the method of separation of variables.

\[
\frac{dy}{dx} = \frac{x}{y} \Rightarrow y \, dy = x \, dx \Rightarrow y^2 - x^2 = \text{constant}
\]

Example 2. Consider the system \( x' = AX \) where

\[
A = \begin{pmatrix}
-1 & -1 \\
1 & -3
\end{pmatrix}
\]

determine the type of stability of the critical point \((0,0)\) and sketch the trajectories passing through the points \( a) x_0 = (1, 0) \)

\( b) x_0 = (0, 1) \)

\[ \det (A - \lambda I) = (\lambda + 2)^2 = 0 \quad \lambda = -2 \quad \text{repeated root} \]

\[
A + 2I = \begin{pmatrix}
1 & -1 \\
1 & -1
\end{pmatrix} \Rightarrow \text{only one eigenvector } K_1 = (1, 1) \Rightarrow K_2 = (0, 1)
\]

The general solution is \( X = c_1 (1, 1)e^{-2t} + c_2 (1, -1)e^{-2t} \)

The origin is an improper node (asymptotically stable).

\( a) \quad X = \begin{pmatrix} 1 + t \\ t \end{pmatrix} e^{-2t} \)

\( b) \quad X = -\begin{pmatrix} 1 + t \\ t \end{pmatrix} e^{-2t} \)
Example 3

\[ X' = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} X \]

Determine the type of critical point at \((0,0)\) and sketch the trajectory passing through the point \(X(0) = (1,0)\).

\[ \det(A-\lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \]

The origin is a center (stable).

Given the initial point \(X(0) = (1,0)\) we get from the general solution

\[ X = c_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix} \]

\[ c_1 = 0, \quad c_2 = -1 \quad \Rightarrow \quad X = \begin{pmatrix} \cos t - \sin t \\ -\sin t \end{pmatrix} \]

Example 4

\[ X' = \begin{pmatrix} 0 & -1 \\ 1 & \epsilon \end{pmatrix} X \quad \epsilon \text{ real } \neq 0 \]

Discuss the type of stability at \((0,0)\)

\[ \det(A-\lambda I) = (\lambda - \epsilon)^2 + 1 \Rightarrow \lambda = \epsilon \pm i \]

If \(\epsilon < 0\), \((0,0)\) is a spiral sink (asymptotically stable)

If \(\epsilon > 0\), \((0,0)\) is a spiral source (unstable)

In particular for \(\epsilon = -\frac{1}{2}\) and \(X(0) = (1,0)\), \(X(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{-\frac{t}{2}}\)
3. Almost linear systems

Consider the system
\[
\begin{align*}
\frac{dx}{dt} &= ax + by + \phi(x, y) \\
\frac{dy}{dt} &= cx + dy + \psi(x, y)
\end{align*}
\]
(A.L.)

where \(a, b, c\) and \(d\) are constants and \(\phi, \psi\) are of class \(C^1\) in a neighborhood of \((0, 0)\). We suppose also that \(\frac{\sqrt{ad - bc}}{\sqrt{cd}} \neq 0\).

In vector notation we can write the above system as
\[X' = AX + G(x)\]
(A.L.)

where \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) and \(G(x) = \begin{pmatrix} \phi(x) \\ \psi(x) \end{pmatrix}\)

Definition: If \(\|G(x)\|/\|x\| \to 0\) as \(\|x\| \to 0\) then the system (A.L.) is called almost linear.

Question: What can we say about the stability or asymptotic stability of the system (A.L.) from what we know of the linear system \(X' = AX\)?

Example: Show that \(\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\epsilon}{2} \end{pmatrix} x - \begin{pmatrix} x^2 + xy \\ \frac{3}{4} xy + \frac{1}{2} y^2 \end{pmatrix}\) is almost linear.

Using polar coordinates, we have
\[
\frac{G(x)}{\|x\|} = -r \begin{pmatrix} 3 \cos \theta + \epsilon \sin \theta \\ \frac{3}{4} \cos \theta \sin \theta + \frac{1}{4} \sin^2 \theta \end{pmatrix} \to 0 \quad \text{as} \quad r \to 0.
\]

\(:\) The system is almost linear.
Example 2. The scalar pendulum equation is originally defined by

$$\frac{d^2 \theta}{dt^2} + c \frac{d\theta}{dt} + \omega^2 \sin \theta = 0$$

where \( c \) is the damping coefficient and \( \omega = \sqrt{g/L} \)

If we let \( x = \theta \), \( y = \frac{d\theta}{dt} \), then we obtain the system

$$\begin{cases}  
x' = y \\
y' = -\omega^2 \sin x - cy \end{cases} \quad (P)$$

Find the critical points of the system \((P)\) and show that the system is almost linear near the origin.

We should have

$$\begin{pmatrix} y \\ \omega^2 \sin x + cy \end{pmatrix} = 0 \Rightarrow \begin{cases} y = 0 \\
\sin x = 0 \end{cases} \Rightarrow (x = 0, \Pi, 2\Pi, \ldots)$$

The critical points are: \((0,0)\), \((\Pi,0)\), \((2\Pi,0)\),...

To show that the system is almost linear, we first write \( \sin x = x + (\sin x - x) \). Therefore

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 - c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \omega^2 \begin{pmatrix} 0 \\ x - \sin x \end{pmatrix}$$

Using the expansion \( \sin x = x - \frac{x^3}{3!} + \ldots \)

we get, using polar coordinates system, \( \frac{x - \sin x}{r} \to 0 \) as \( r \to 0 \)

Therefore \((P)\) is almost linear near \((0,0)\)
Example 3. Find the linear system of the pendulum equation 

\[ (P) \text{ near the point } (\pi, 0) \]

Using Taylor formula at the point \( P_0 = (x_0, y_0) \) we get:

\[
\begin{align*}
    f(x, y) &= f(P_0) + (P - P_0) \cdot Df(P_0) + \ldots \\
    g(x, y) &= g(P_0) + (P - P_0) \cdot Dg(P_0) + \ldots
\end{align*}
\]

At the critical point \( P_0 = (\pi, 0) \), \( f(P_0) = 0 = g(P_0) \) and

\[
\begin{align*}
    f_x(P_0) &= 0, \\
    f_y(P_0) &= 1, \\
    g_x(P_0) &= -\omega^2 \cos x \bigg|_{x = \pi} = -\omega^2, \\
    g_y(P_0) &= -c
\end{align*}
\]

Therefore

\[
\frac{d}{dk} \begin{pmatrix} x - \pi \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -c \end{pmatrix} \begin{pmatrix} x - \pi \\ y \end{pmatrix}
\]

At \( (\pi, 0) \), \( \lambda = -\frac{c}{2} \pm \sqrt{\omega^2 \cdot \frac{c^2}{4}} \Rightarrow 2 \text{ real roots of opposite sign. Therefore } (\pi, 0) \text{ is a saddle point, unstable.} \)

Example 4. The purpose of this example is to show that a "center" for the linear system is not necessarily a "center" for the almost linear system.
We consider the system

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix} - \begin{pmatrix}
  x(x^2+y^2) \\
  y(x^2+y^2)
\end{pmatrix} \quad \text{(N.L.)}
\]

\((0,0)\) is clearly a center for the linear system

\[x' = \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} x.\] Since \((x',y') \to 0\) as \(r \to 0\), \,(N.L.)
is almost linear. Using polar coordinates system, we get

\[r^2 = x^2 + y^2, \quad \theta = \arctan \frac{y}{x}.\] Hence

\[
r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} = -r^2 \Rightarrow \frac{dr}{dt} = -r^3
\]

with \(r(0) = r_0 > 0\), we get

\[r = \frac{r_0}{\sqrt{1 + 2r_0 t}} \to 0, \quad t \to \infty\]

Also

\[
\frac{d\theta}{dt} = \frac{x y' - x' y}{x^2 + y^2} = -1 \Rightarrow \theta = -t + \theta_0.
\]

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = \frac{r_0}{\sqrt{1 + 2r_0 t}} \begin{pmatrix}
  \cos(-t + \theta_0) \\
  \sin(-t + \theta_0)
\end{pmatrix}
\]

\((0,0)\) is a spiral point (asym. stable) for (N.L.)

**Note:** If we compare the nullclinal point \((0,0)\) for both the linear system \(x' = Ax\) and (N.L.), it is interesting to note the following facts:
1/ In the following cases there is a similarity of the type of critical point and the type of asymptotic stability or unstability.

\[ \lambda_1 < \lambda_2 < 0 \quad \text{(Node)} \]

\[ 0 < \lambda_1 < \lambda_2 \quad \text{(Node)} \]

\[ \lambda_1 < 0 < \lambda_2 \quad \text{(Saddle point)} \]

\[ \lambda = \alpha + i\beta \quad \begin{cases} 
\alpha < 0 & \text{spiral sink} \\
\alpha > 0 & \text{spiral source} 
\end{cases} \]

2/ Changes may occur in the following cases

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<td>b) ( \lambda_1 = \lambda_2 &gt; 0 )</td>
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<tr>
<td>c) ( \lambda = \pm i\beta )</td>
<td>Center</td>
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4. Applications: The Predator-Prey equations

The following predator-prey equations

\[
\begin{align*}
\frac{dx}{dt} &= ax - \kappa xy \\
\frac{dy}{dt} &= -cy + \beta xy
\end{align*}
\]

were first introduced based on simple observations of animal life. Both Lotka and Volterra made significant contributions to the formulation and results of the problem.

Let \( x(t) \) represent the rabbits (prey) population and \( y(t) \) the foxes (predator) population. In the absence of foxes the rabbits grow at a rate proportional to its current population \( \frac{dx}{dt} = ax \) \,(exponential growth)\,

But in the absence of rabbits the foxes die out \( \frac{dy}{dt} = -cy \) \,(exponential decay)\,

However we are interested in situations where \( 0 < x, y < \infty \)
According to the model, the growth rate of the rabbits is
decreased by a term \( ax \) (a quantity proportional to
the product of their population), while the growth rate of
the foxes is increased by a term \( bx \).

Example.

\[
\begin{align*}
\frac{dx}{dt} &= x - \frac{1}{2} xy \\
\frac{dy}{dt} &= -\frac{3}{4} y + \frac{1}{4} xy
\end{align*}
\]

The critical points are given by
\[
\begin{pmatrix}
x(1-\frac{1}{2}y)
\gamma(-\frac{3}{4} + \frac{1}{4} x)
\end{pmatrix} = 0
\]

\(\Rightarrow (0,0)\) and \((3,2)\) are the critical points of the
system. For the point \((0,0)\), the linear system
is
\[
\begin{pmatrix}
x
\gamma
\end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
\det(A - \lambda I) = (\lambda + \frac{3}{4})(\lambda - 1) = 0 \Rightarrow \lambda = 1, \lambda = -\frac{3}{4}
\]

\((0,0)\) is a saddle point (unstable)

\[
x = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{3}{4} t}
\]
For the point $(3, 2)$, the linear system is

\[ \frac{d}{dt} \begin{pmatrix} x-3 \\ y-2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x-3 \\ y-2 \end{pmatrix} \]

\[ \det (A - \lambda I) = \lambda^2 + \frac{3}{4} = 0 \Rightarrow \lambda = \pm i \frac{\sqrt{3}}{2} \]

Therefore $(3, 2)$ is a center for the linear system (stable).

If \( \begin{cases} u = x-3 \\ v = y-2 \end{cases} \) then \( \frac{dv}{du} = \frac{dv/dt}{du/dt} = -\frac{\frac{1}{2}u}{\frac{3}{2}v} = -\frac{u}{3v} \)

Therefore \( u \frac{du}{dv} + 3v \frac{dv}{dv} = 0 \Rightarrow u^2 + 3v^2 = \text{constant} \Rightarrow \) family of ellipses.

To get a more precise information about the trajectories, we can go back to the original system and obtain

\[ \frac{dy}{dx} = \frac{y\left(-\frac{3}{4} + \frac{1}{4}x\right)}{x\left(1-\frac{1}{2}y\right)} \]

which gives,

by the method of separation of variables

\[ \frac{1}{2} y - \ln y + \frac{1}{4} x - \frac{3}{4} \ln x = \text{constant} \]

Using plotting tools for the trajectories, it can be shown that $(3, 2)$ is a center (and stable critical point) for the nonlinear system and has cyclic time variation for $x$ and $y$. 