MATH 2400  Introduction to differential equations
Sections 5–8 and 13–20
Homework #3 Solutions

Hw #3 is:

Section 3.1:   # 2, 18, 21
Section 3.2:   # 1, 11
Section 3.3:   # 5, 18
Section 3.4:   # 7, 20, 11

Section 3.1  Second order linear equations with constant coefficients. Real and distinct roots of the characteristic equation.

#2. Find the general solution of the differential equation:

\[ y'' + 3y' + 2y = 0 \]

For solving (1), assume (*) \( y = e^{rt} \).

By plugging (*) in (1), we obtain the characteristic equation:

\[ r^2 + 3r + 2 = 0 \]
For (2), \[ \Delta = b^2 - 4ac = 9 - 8 = 1 \]

\[ \lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-3 \pm 1}{2} \]

\[ \Rightarrow \lambda_1 = \frac{-3 + 1}{2} = -1, \quad \lambda_2 = \frac{-3 - 1}{2} = -2 \]

are the roots of the characteristic equation (2)

Therefore, two particular solutions of (1) are

\[ y_1(t) = e^{-t} \quad \text{and} \quad y_2(t) = e^{-2t} \]

and the general solution of (1) is:

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-t} + c_2 e^{-2t} \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

# 18. Find a differential equation whose general solution is:

\[ y(t) = c_1 e^{-t/2} + c_2 e^{-2t} \]

From the solution given above, we recognize that the roots of the characteristic equation are \( \lambda_1 = -\frac{1}{2} \) and \( \lambda_2 = -2 \), and therefore the characteristic equation is:
\[(\lambda + \frac{1}{2})(\lambda + 2) = 0 \iff \lambda^2 + \frac{5}{2}\lambda + 1 = 0 \]

\[
\iff \lambda^2 + \frac{5}{2}\lambda + 1 = 0 \quad \therefore 2
\]

\[
\iff (3) \quad 2\lambda^2 + 5\lambda + 2 = 0
\]

Since the coefficients of the characteristic equation are identical with the coefficients of the differential equation, and vice-versa, we recognize that the characteristic equation \((3)\) corresponds to the differential equation:

\[2y'' + 5y' + 2y = 0.\]

Solve the initial value problem

\[(4) \quad y'' - y' - 2y = 0\]

\[(5) \quad y(0) = \lambda, \quad y'(0) = 2.\]

Then find \(\lambda\) so that the solution approaches zero as \(t \to \infty\).

Let us start with solving the initial value problem (IVP, \(4\)) - (5).

Assuming \(y = e^{\lambda t}\),
we obtain the characteristic equation:

\[ r^2 - r - 2 = 0 \]

\[ \Delta = b^2 - 4ac = 1 + 8 = 9 \]

\[ r_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{1 \pm 3}{2} \]

\[ r_1 = 2, \quad r_2 = -1 \]

Therefore, the general solution of (4) is:

\[ y(t) = c_1 e^{2t} + c_2 e^{-t} \]

with \( c_1 \) and \( c_2 \) arbitrary constants. We use the initial conditions (5) to determine \( c_1 \) and \( c_2 \).

For this, we compute \( y'(t) = 2c_1 e^{2t} - c_2 e^{-t} \).

Therefore:

\[ \begin{align*}
    y(0) &= \alpha \\
    y'(0) &= 2
\end{align*} \]

\[ \begin{cases} 
    c_1 + c_2 = \alpha \\
    2c_1 - c_2 = 2
\end{cases} \]

Adding the two equations eliminates \( c_2 \) to get:

\[ 3c_1 = \alpha + 2 \quad \Rightarrow \quad c_1 = \frac{\alpha + 2}{3} \]

Substitute this in the 1st equation to obtain

\[ c_2 = \alpha - c_1 = \alpha - \frac{\alpha + 2}{3} = \frac{2\alpha - 2}{3} = \frac{2(\alpha - 1)}{3} \]
Therefore, the solution of the (IVP) (1)-(5) is:

\[ y(t) = \frac{x+2}{3} e^{2t} + \frac{2(x-1)}{3} e^{-t} \]

For \( y \to 0 \) as \( t \to \infty \), we need to equilibrate the first term in the sum above, therefore we take

\[ \frac{x+2}{3} = 0 \implies x = -2. \]

3.2. Fundamental solutions of linear homogeneous equations.

The Wronskian.

#1

Find the Wronskian of the given pair of functions:

\[ f(t) = e^{2t}, \quad g(t) = e^{-3t/2}. \]

For two functions \( f(t) \) and \( g(t) \), by definition the Wronskian is given by (6) \( W(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} \).

In our case:

\[ W(t) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2} e^{-3t/2} \end{vmatrix} = \]

\[ = -\frac{3}{2} e^{2t} - 2 e^{2t} - \frac{3}{2} e^{\frac{-3t}{2}} = -\frac{7}{2} e^{\frac{2t}{2}} = \]

\[ = -\frac{7}{2} e^{\frac{t}{2}} \]
Determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution for:

(7) \((x - 3) y'' + x y' + (\ln |x|) y = 0\)

(8) \(y(1) = 0\), \(y'(1) = 1\)

Using the uniqueness and existence theorem 3.2.1 on page 144 in the textbook, we impose the continuity of the functions \(p(x), q(x)\) and \(g(x)\) in the standard form:

(9) \(y'' + p(x) y' + q(x) y = g(x)\)

Bringing (7) to the standard form (9) by dividing (7) with \((x - 3)\), we obtain:

\[
y'' + \frac{x}{\ln |x|} y' + \frac{\ln |x|}{x - 3} y = 0.
\]

Therefore, for our case: \(p(x) = \frac{x}{x - 3}\), \(q(x) = \frac{\ln |x|}{x - 3}\)

and \(g(x) = 0\).

For \(q(x)\) to be
\( p(x) \) is discontinuous at \( x = 3 \) only.

\( q(x) \) is discontinuous at \( x = 3 \) and \( x = 0 \) only.

Since the initial conditions (8) are given for \( x_0 = 1 \), then a solution of (7)-(8) exists and is unique in the interval \( I = (0, 3) \), which contains the initial data \( x_0 = 1 \).

3.3. Linear independence and the Wronskian

#5. In the following problem, determine whether the given pair of functions is linearly independent or linearly dependent:

\[ f(t) = 3t - 5, \quad g(t) = 9t - 15. \]

According to the definition on page 153, two functions \( f(t) \) and \( g(t) \) are linearly dependent if there exists two constants \( k_1 \) and \( k_2 \), not both zero, such that (10) \( k_1 f(t) + k_2 g(t) = 0 \) for all \( t \in I \) and they are linearly independent if (10) holds for all \( t \in I \) only when \( k_1 = k_2 = 0 \).
In our case, since
\[ g(t) = 9t - 15 = 3 \cdot (3t - 5) = 3 \cdot f(t) \Rightarrow g(t) - 3f(t) = 0, \]
for all \( t \in \mathbb{I} \Rightarrow (10) \) is verified with the functions \( f(t) \) and \( g(t) \)
are linearly dependent.

Find the Wronskian of two solutions of the differential equation:
\[ (11) \quad (1 - x^2) y'' - 2x y' + a(x+1) y = 0 \]
(Legendre's equation)
without solving the equation.

According to Abel's theorem (Theorem 3.3.2 on page 155 in the textbook), for an equation
\[ y'' + p(x) y' + q(x) y = 0 \]
the Wronskian is given by:
\[ W(y_1, y_2)(x) = C e^{-\int p(x) dx} \]
where \( C \) is an arbitrary constant on \( y_1 \) and \( y_2 \) only.

In our case:
\[ (11) \Rightarrow y'' - \frac{2x}{1 - x^2} y' + a(x+1) \frac{y}{1 - x^2} = 0 \]
\[ \int \frac{2x}{1 - x^2} dx \]
\[ \Rightarrow W(y_1, y_2)(x) = C e \]
\[
\int \frac{2x}{1-x^2} \, dx \quad \text{is solved using a change of variables:} \quad u = 1-x^2 \quad \Rightarrow \quad du = -2x \, dx
\]
\[
\Rightarrow \quad \int \frac{2x}{1-x^2} \, dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|1-x^2| + C
\]

Then \( W(y_1, y_2)(x) = Ce^{-\ln|1-x^2|} = Ce^{\ln|1-x^2|^{-1}} = \frac{1}{|1-x^2|} \)

3.4. Complex roots of the characteristic equation

#1. Find the general solution of the given differential equation:

(12) \( y'' - 2y' + 2y = 0 \)

Assume that \( y = e^{rt} \), then we obtain the characteristic equation:

\( r^2 - 2r + 2 = 0 \)

\( \Delta = b^2 - 4ac = 4 - 8 = -4 \)

\( r_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i \)
Therefore, two particular solutions are $y(12)$ are

$$y_1(t) = e^{2t} = e^{(1+i)t} = e^t (\cos t + i \sin t)$$

and

$$y_2(t) = e^{2t} = e^{(1-i)t} = e^t (\cos t - i \sin t)$$

From $y_1(t)$ and $y_2(t)$ above, we obtain that a pair of real-valued solutions is

$$y_1(t) = e^t \cos t \quad \text{and} \quad y_2(t) = e^t \sin t$$

Therefore, the general solution $y(12)$ is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^t \cos t + c_2 e^t \sin t$$

where $c_1$ and $c_2$ are two arbitrary constants.

#20. Find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing $t$.

(13) $y'' + y = 0$

(14) $y(\frac{\pi}{3}) = 2 \quad y'(\frac{\pi}{3}) = -3$

The characteristic equation for (13) is

$$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$$

are the roots of the characteristic equation.
Two particular solutions of (13) are:

\[ y_1(t) = e^t (\cos t + i \sin t) = \cos t + i \sin t \]

and

\[ y_2(t) = e^t (\cos t - i \sin t) = \cos t - i \sin t \]

and two real valued solutions are

\[ y_1(t) = \cos t \quad \text{and} \quad y_2(t) = \sin t \]

Therefore the general solution of (13) is:

\[ y(t) = C_1 \cos t + C_2 \sin t \]

We have \( y'(t) = -C_1 \sin t + C_2 \cos t \)

Therefore (14) gives:

\[
\begin{align*}
    y(\frac{\pi}{2}) &= 2 \\
y'(\frac{\pi}{2}) &= -4
\end{align*}
\]

\[
\begin{align*}
    C_1 \cos \frac{\pi}{3} + C_2 \sin \frac{\pi}{3} &= 2 \\
    -C_1 \sin \frac{\pi}{3} + C_2 \cos \frac{\pi}{3} &= -4
\end{align*}
\]

\[
\begin{align*}
    \frac{C_1}{2} + \frac{C_2}{2} &= 1 \\
    -\frac{C_1}{2} \sqrt{3} + \frac{C_2}{2} &= -2
\end{align*}
\]

\[
\begin{align*}
    C_1 + \sqrt{3} C_2 &= 4 \\
    -\sqrt{3} C_1 + C_2 &= -8
\end{align*}
\]

\[
\begin{align*}
    4 C_2 &= 4\sqrt{3} - 8 \quad \Rightarrow \quad C_2 = \sqrt{3} - 2 \\
    C_1 &= 4 - \sqrt{3} C_2 = 4 - \sqrt{3} (\sqrt{3} - 2) = 4 - 3 + 2\sqrt{3} = 1 + 2\sqrt{3}
\end{align*}
\]
Therefore the solution of the (IVP) (13) - (14) is:

\[ y(t) = (1 + 2\sqrt{3}) \cos t + (\sqrt{3} - 2) \sin t \]

Since this solution does not have any exponential factor, it is a steady oscillation for all \( t \).

Considering the "initial conditions"

\[ y\left(\frac{\pi}{5}\right) = 2, \quad y'(\frac{\pi}{5}) = -4 \]

we can sketch the graph of \( y(t) \):

As \( t \to \infty \), \( y \) oscillates.

Find the general solution of the differential equation

\[ (15) \ y'' + 6y' + 13y = 0 \]

Assuming \( y = e^{rt} \), produces the characteristic equation

\[ (16) \ r^2 + 6r + 13 = 0 \]
For (16), we have: \( \Delta = b^2 - 4ac = 36 - 52 = -16 \)

Then \( x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i \)

Therefore, two particular solutions of (15) are:

\[ y_1(t) = e^{-3t} \left( \cos 2t + i \sin 2t \right) \quad \text{and} \]

\[ y_2(t) = e^{-3t} \left( \cos 2t - i \sin 2t \right) \]

and two real valued solutions are:

\[ y_1(t) = e^{-3t} \cos 2t \quad \text{and} \quad y_2(t) = e^{-3t} \sin 2t \]

The general solution of the differential equation (15) is:

\[ y(t) = c_1 e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t \]