HW #10, Week of November 14

10.5 #8, 12, 23
10.6 #1
10.7 # 3a, 5a, 13
10.8 # 2, 5, 8a & 6

10.5 Separation of Variables: Heat Conduction in a Rod

p610. #8. Find the solution of the heat conduction problem:

\[ \frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0 \]

\[ u(x, 0) = 0 \]

\[ u(2, t) = 0 \quad \forall \quad t > 0 \]

\[ u(x, t) = 2 \sin \left( \frac{\pi x}{2} \right) - \sin \pi x + 4 \sin 2 \pi x \]

\[ \forall \quad 0 \leq x \leq 2 \]

- \( \frac{\partial^2 u}{\partial x^2} - 4 \frac{\partial u}{\partial t} = 0 \); \( u(0, t) = u(2, t) = 0 \)

so heat conduction eqn. and the boundary conditions are homogeneous.

- First, put the equation in standard form:

\[ \frac{d^2}{dt^2} \left( \frac{1}{4} \frac{\partial^2 u}{\partial x^2} \right) = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0 \]

\[ x = \frac{1}{2} \]

\[ u(0, t) = 0 = u(2, t) \]

\[ L = 2 \]

\[ u(x, 0) = f(x) = 2 \sin \left( \frac{\pi x}{2} \right) - \sin \pi x + 4 \sin 2 \pi x \]

\[ \frac{d^2}{dt^2} \left( \frac{1}{4} \sin \frac{\pi x}{2} \right) = -\left( \frac{\pi^2}{4} \right) t - \frac{\pi^2}{4} \]

use #19, p607, to obtain the general solution:

\[ u = \sum_{n=1}^{\infty} c_n e^{-\frac{(n \pi t)^2}{16}} \sin \frac{n \pi x}{2} \]

\[ \Rightarrow 2 \sin \frac{\pi x}{2} - \sin \pi x + 4 \sin 2 \pi x = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n \pi x}{2} \]

- Fourier sine series for \( f(x) \)
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8) (continued) equating coefficients, we get:

\[ c_1 = 2, \quad c_2 = -1, \quad c_4 = 4 \]

\[ \therefore \quad u(x, t) = 2 \sin \frac{\pi}{2} x e^{\frac{\pi^2}{16} t} - \sin \pi x e^{\frac{\pi^2}{16} t} + 4 \sin 2\pi x e^{\frac{\pi^2}{16} t} - \frac{16 \pi^2}{16} t \]

\[ \therefore \quad = 2 \sin \frac{\pi}{2} x e^{\frac{\pi^2}{16} t} - \sin \pi x e^{\frac{\pi^2}{4} t} + 4 \sin 2\pi x e^{\frac{\pi^2}{4} t} - \pi^2 t \]
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10.5 #12. Consider the conduction of heat in a rod 40 cm long, whose ends are maintained at 0°C for all \( t > 0 \).

Find an expression for the temperature \( u(x,t) \) if the initial temperature distribution in the rod is

\[
u(x,0) = x \quad 0 < x < 40.
\]

Assume \( a^2 = 1 \) in \( a^2 u_{xx} = u_t \) (p. 605)

The heat conduction problem is formulated as:

\[
u_{xx} = u_t \quad 0 < x < 40, \quad t > 0
\]

\[
u(0,t) = 0 \quad \nu(40,t) = 0 \quad t > 0
\]

\[
u(x,0) = x \quad 0 < x < 40
\]

Following the method in problem #8, we consider

\[
\frac{X''}{X} = \frac{T'}{T} = -\lambda
\]

subject to

\[
X(0) = X(40) = 0
\]

per p. 605, nontrivial solutions have the form

\[
X_n(x) = \sin \left( \frac{n\pi x}{L} \right) \quad n = 1, 2, 3
\]

with eigenvalues

\[
\lambda_n = \left( \frac{n\pi}{L} \right)^2 \quad n = 1, 2, 3
\]

so eigenfunctions for this problem are:

\[
X_n = \sin \left( \frac{n\pi x}{40} \right) \quad \text{eigenvalues} \quad \lambda_n = \left( \frac{n\pi}{40} \right)^2
\]
The solutions of the temporal equation,
\[ T'' + \lambda T = 0 \]
are
\[ T_n = e^{-\frac{\lambda_n t}{2}} = e^{-\left(\frac{n\pi}{40}\right)^2 t} \]

so our general solution is
\[ u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin\left(\frac{n\pi x}{40}\right) \]

The coefficients \( c_n \) are the Fourier sine coeffs of \( u(x,0) = x \) (read pp 607-608)

\[ \Rightarrow c_n = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{40}\right) dx \]

\[ \frac{1}{20} \left[ \int_0^{40} x \sin\left(\frac{n\pi x}{40}\right) dx \right]^{40} \]

\[ \Rightarrow -\frac{2}{n\pi} \left(\frac{40}{\cos(n\pi)}\right) \cos n\pi + \frac{1}{20}\left(\frac{40}{\pi}\right)^2 \sin\left(\frac{n\pi x}{40}\right) \right]_{0}^{40} \]

\[ \Rightarrow -\frac{80}{n\pi} \cos n\pi \]
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#12 finish:) \[-80 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \]

Thus \[u(x,t) = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin \left(\frac{n\pi}{40}\right) x \]

10.5 #23: The heat conduction equation in two space dimensions may be expressed in terms of polar coordinates as:

\[a^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right] = \frac{\partial u}{\partial t} \]

Assuming that \[u(r,\theta,t) = R(r) \Theta(\theta) T(t)\]

find ordinary differential equations that are satisfied by \[R(r), \Theta(\theta), \text{and } T(t)\]

- Substituting \[u(r,\theta,t)\] into the PDE:

\[a^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial R \Theta T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 R \Theta T}{\partial \theta^2} \right] = R \Theta T' \]

- Divide both sides by \[R \Theta T\]; set \[-\lambda^2\]

\[\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \Theta'' = T'' \]

- First consider:

\[\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \Theta'' = -\lambda^2 \]

\[\Rightarrow \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \Theta'' = -\lambda^2 \]

Multiply both sides by \[r^2\]; rearrange
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23 (continued):

\[ \frac{r^2 R''}{R} + \frac{r R'}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = \mu^2 \]

Can use a second separation constant here.

\[ \frac{r^2 R''}{R} + \frac{r R'}{R} + \lambda^2 r^2 = \mu^2 \]

\[-\frac{\Theta''}{\Theta} = \mu^2 ; \quad \text{recall } \frac{T'}{aT} = -\lambda^2 \]

Resulting ODEs are

\[ r^2 R'' + r R' + \left( \lambda^2 r^2 - \mu^2 \right) R = 0 \]

\[ \Theta'' + \mu^2 \Theta = 0 \]

\[ T' + a^2 \lambda^2 T = 0 \]

10.6 Other Heat-Conduction Problems

p620 #1. Find the steady-state solution of the heat-conduction equation \( \partial^2 u / \partial x^2 = \partial t \) that satisfies \( u(0, t) = 10 \), \( u(50, t) = 40 \)

Using equation 16 (p. 614), the general solution:

\[ u(x, t) = (T_2 - T_1) \frac{x}{L} + T_1 + \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n \pi x}{L}\right)^2 t} \sin \left(\frac{n \pi x}{L}\right) \]
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10.6 #1 (continued)

of the problem $a^2 u_{xx} = u_t$,

$t \to \infty$

$u(x,0) = f(x)$
$(u(x,t)) = \begin{cases} T_1 & \text{if } x < \frac{L}{2} \\ T_2 & \text{if } x \geq \frac{L}{2} \end{cases}$

$u = \left( T_2 - T_1 \right) \frac{x}{L} + T_1 \quad \text{as } t \to \infty \quad \text{(steady-state soln)}.$

For this problem $L = 50$, $T_2 = 40$, $T_1 = 10$

$\Rightarrow 30 \left( \frac{x}{50} \right) + 10 \Rightarrow u = \frac{3}{5} x + 10.$

10.7: The Wave Equation - Elastic String Vibrations

p626 Consider an elastic string of length $L$ with fixed ends. The string is set in motion with no initial velocity from an initial position $u(x,0) = f(x)$

#3a) $f(x) = 8x (L-x)^2 / L^3$; find the displacement $u(x,t)$

The initial velocity is zero. Per equation 20, p626, we assume a solution of the form

$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi at}{L} \right)$

where the $c_n$ are the Fourier sine coefficients of $f(x)$.

$\Rightarrow c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$
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10.7 #3a continued

\[ c_n = 2 \frac{\pi}{L} \int_0^L 8x(L-x)^2 \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) dx \]

\[ = \frac{16}{L^4} \int_0^L x(L-x)^2 \sin \left( \frac{n\pi x}{L} \right) dx \]

\[ = \left( \frac{2}{L} \right)^4 \int_0^L \left[ x^3 - 2x^2 L + L^2 x \right] \sin \left( \frac{n\pi x}{L} \right) dx \]

\[ = \left( \frac{2}{L} \right)^4 \left[ \int_0^L x^3 \sin \left( \frac{n\pi x}{L} \right) dx - \int_0^L 2x^2 L \sin \left( \frac{n\pi x}{L} \right) dx + \int_0^L L^2 x \sin \left( \frac{n\pi x}{L} \right) dx \right] \]

\[ = 32 \left[ \frac{2 + \cos n\pi}{(n\pi)^3} \right] = C_n \]

\[ u(x,t) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{2 + \cos n\pi}{n^3} \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi t}{L} \right) \]
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#5a) Now the string is set in motion from equilibrium (initial displacement = 0) with an initial velocity

\[ u_t(x,0) = g(x) = \begin{cases} \frac{2x}{L} & 0 \leq x \leq \frac{L}{2} \\ \frac{2(L-x)}{L} & \frac{L}{2} < x < L \end{cases} \]

Find the displacement \( u(x,t) \).

Since initial displacement is now = 0, we seek a solution of the form

\( \rho 63 (\#34) \):

\[ u(x,t) = \sum_{n=1}^{\infty} k_n \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \]

where \( k_n \) are the Fourier sine coefficients of \( u_t(x,0) = g(x) \)

Thus

\[ k_n = \frac{2}{\rho \pi a} \int_{0}^{L} g(x) \sin \frac{n\pi x}{L} \, dx \]

\[ \Rightarrow k_n = \frac{2}{\rho \pi a} \left[ \int_{0}^{L/2} \frac{2x}{L} \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^{L} \frac{2(L-x)}{L} \sin \frac{n\pi x}{L} \, dx \right] \]

\[ \Rightarrow \frac{2}{\rho \pi a} \left[ \frac{2}{L} \int_{0}^{L/2} x \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^{L} 2 \sin \frac{n\pi x}{L} \, dx \right] = \frac{8L \sin \frac{\pi}{2}}{(\pi \pi)^{3a}} \]
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*5a finish*) displacement is then given by

$$w(x,t) = \frac{8L}{a\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3 \sin \frac{\pi n}{L} \sin \frac{n\pi x}{L}} \sin \frac{n\pi t}{L}$$

10.7

p634 #13) show that the wave equation

$$a^2 u_{xx} = u_{tt}$$

reduces to $u\varphi\eta = 0$

using the change of variables

$$\xi = x - at, \ \eta = x + at$$

Compute partial derivatives using the Chain Rule:

$$u_t = U_\xi \xi_t + U_\eta \eta_t = a U_\eta - a U_\xi$$

$$u_{tt} = a^2 (U_{\xi\xi} + U_{\eta\eta}) - (U_{\xi\xi} \xi_t + U_{\xi\eta} \eta_t)$$

$$= a^2 (U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta})$$

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta$$

$$u_{xx} = U_{\xi\xi} \xi_x + U_{\xi\eta} \eta_x + U_{\xi\eta} \xi_x + U_{\eta\eta} \eta_x$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Combining, we find:

$$u_{tt} - a^2 u_{xx} = a^2 (U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}) - a^2 (U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta})$$

$$= -4a^2 U_{\xi\eta}$$

We get immediate that $u_{tt} - a^2 u_{xx} = 0$ implies

$$-4a^2 U_{\xi\eta} = 0 \Rightarrow U_{\xi\eta} = 0$$
10.8 Laplace's Equation \( u_{xx} + u_{yy} = 0 \)

p 644 2) Find the solution \( u(x,y) \) of Laplace's equation in the rectangle \( 0 < x < a \)

\( 0 < y < b \)

that satisfies the boundary conditions:

\( u(0,y) = 0 \quad u(a,y) = 0 \quad 0 < y < b \)

\( u(x,0) = h(x) \quad u(x,b) = 0 \quad 0 \leq x \leq a \)

Using \( u(x,y) = X(x)Y(y) \) and separation of variables returns:

\[ X''Y + Y''X = 0 \]

\[ \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \]

\( i) \Rightarrow X'' + \lambda X = 0, \quad X(0) = X(a) = 0 \)

and

\( ii) \quad Y'' - \lambda Y = 0, \quad Y(b) = 0 \)

For case \( i) \), we again refer to p. 607, nontrivial solutions will have the form:

\[ X_n(x) = \sin \left( \frac{n\pi x}{a} \right) \quad n = 1, 2, 3, \]

with eigenvalues \( \lambda_n = \left( \frac{n\pi}{a} \right)^2 \)

For case \( ii) \) form 31 p 611 (c gives us)

\[ Y(y) = d_1 \cosh \frac{n\pi}{a} \left( b - y \right) + d_2 \sinh \frac{n\pi}{a} \left( b - y \right) \]

NB: we use this form to satisfy the bc's for \( y = 0, y = b \) namely \( u(x,0) = h(x) \), \( u(x,b) = 0 \)
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10.8 #2 (continued)

for \( Y(y) = d_1 \cosh \sqrt{\lambda} (b-y) + d_2 \sinh \sqrt{\lambda} (b-y) \)

Impose our b.c. \( Y(b) = 0 \)

\[ \Rightarrow Y(b) = d_1 \cosh (0) + d_2 \sinh (0) = 0 \]

\[ \Rightarrow d_1 = 0 \quad \Rightarrow Y(y) = d_2 \sinh \sqrt{\lambda} (b-y) \]

Thus
\[ u_n(x,y) = X \cdot Y = \sin \left( \frac{n\pi x}{a} \right) \sinh \frac{n\pi (b-y)}{a} \]

and
\[ u(x,y) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{a} \right) \sinh \frac{n\pi (b-y)}{a} \]

The coefficients are determined by the boundary condition \( u(x,0) = h(x) \)

\[ \Rightarrow \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi b}{a} \sinh \frac{n\pi b}{a} = h(x) \]

Coefficients are calculated using the equation
\[ c_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_{0}^{a} h(x) \sin \frac{n\pi x}{a} \, dx \]

\[ \Rightarrow c_n = \frac{4/a}{\sinh(\pi b/a)} \int_{0}^{a} h(x) \sin \frac{n\pi x}{a} \, dx \]
10.8 #5) Find the solution \( u(r, \theta) \) of Laplace's Equation

\[
\nabla u + \left( \frac{1}{r} \right) u_r + \left( \frac{1}{r^2} \right) u_{\theta\theta} = 0
\]

outside the circle \( r = a \), satisfying the boundary condition \( u(a, \theta) = f(\theta) \) \( 0 \leq \theta < 2\pi \) on the circle.

Assume \( u(r, \theta) \) is single-valued and odd for \( r > a \).

The general solution form for this type of equation for \( r > a \) is:

\[
u = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\theta) + b_n \sin(n\theta) \right] \left( \frac{r}{a} \right)^{-n}
\]

\[
u = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\theta) + b_n \sin(n\theta) \right] \left( \frac{a}{r} \right)^n
\]

NB: we must choose \( \left( \frac{r}{a} \right)^{-n} = \left( \frac{a}{r} \right)^n \) since \( r \in (a, \infty) \) and we want our solution to be bounded.

where

\[
a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) \, d\theta
\]

\[
b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) \, d\theta
\]
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8a) Find the solution $u(x, y)$ of Laplace's eqn in the semi-infinite strip $0 < x < a$, $y > 0$, that satisfies the b.c.'s

\[ u(0, y) = 0 \quad u(a, y) = 0 \quad y > 0 \]
\[ u(x, 0) = f(x) \quad 0 \leq x \leq a \]

and the additional condition $u(x, y) \to 0$ as $y \to \infty$.

b) Find the solution if $f(x) = x(a - x)$

a) For this solution, we must choose the form

\[ u = \sum_{n=1}^{\infty} b_n e^{-\frac{n\pi}{a} y} \sin \left( \frac{n\pi}{a} x \right) \]

picking $-\frac{n\pi}{a} y$ in the exponent since $y \in (0, \infty)$ and we are, again, seeking a bounded solution, and:

\[ b_n = \frac{2}{a} \int_0^a f(x) \sin \left( \frac{n\pi}{a} x \right) \, dx \]

b) When $f(x) = x(a - x)$, we must set up integration by parts

\[ b_n = \frac{2}{a} \left[ \left( xa - x^2 \right) \cos \frac{n\pi}{a} x \right]_0^a - \frac{a}{n\pi} \int_0^a \left( a - 2x \right) \cos \frac{n\pi}{a} x \, dx \]

\[ = \frac{2}{a} \left[ \left( xa - x^2 \right) \cos \frac{n\pi}{a} x \right]_0^a - \frac{a}{n\pi} \left( a - 2x \right) \sin \frac{n\pi}{a} x \right]_0^a \]

\[ = \frac{2}{a} \left[ \left( xa - x^2 \right) \cos \frac{n\pi}{a} x \right]_0^a - \frac{2a}{n\pi} \left( a - 2x \right) \sin \frac{n\pi}{a} x \right]_0^a \]
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8b (continued):

\[ \frac{2}{a} \left( \frac{a}{n \pi} \right)^a (a-2x) \cos \left( \frac{n \pi}{a} x \right) dx \]

\[ \Rightarrow \frac{2}{a} \left( \frac{a}{n \pi} \right)^a \left[ (a-2x) \sin \left( \frac{n \pi}{a} x \right) \right]_0^a + \frac{2}{n \pi} \left[ \sin \left( \frac{n \pi}{a} x \right) dx \right]_0^a \]

\[ \Rightarrow \frac{4/a}{(n \pi)^3} \left[ - \cos \left( \frac{n \pi}{a} x \right) \right]_0^a \]

\[ = \frac{4a^2}{(n \pi)^3} \left[ 1 - \cos (n \pi) \right] \]