A

Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

Third Exam, Friday, December 9, 2005.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts one hundred and ten minutes.

SOLUTIONS

Q1		/ 40
Q2		/ 10
Q3		/ 20
Q4		/ 15
Q5		/ 15
Total		/ 100
Grade		

1. (40 points) Consider the standard form LP

$$\begin{array}{llrcrcrcl} \min & 10x_1 & + & x_2 & & & & & \\ \text{s.t.} & x_1 & - & x_2 & + & 2x_3 & = & 3 \\ & & & x_2 & - & x_3 & = & 2 \\ & & & & x_1, x_2, x_3 & & \geq & 0 \end{array}$$

   (a) (7 points) Show that $x^{(0)} = (4, 3, 1)$ is an appropriate point to start the affine scaling algorithm.

   (b) (6 points) Derive the associated scaled standard form corresponding to the solution $x^{(0)}$.

   (c) (7 points) The move direction for the affine scaling algorithm is $\Delta x = 7.669(-1, 1, 1)$. Show that this direction is improving and feasible at $x^{(0)}$.

   (d) (6 points) Compute the **maximum possible steplength** $\lambda_{\max}$ in the direction Δx.

   (e) (7 points) For this problem, why must a log barrier method give a direction from $x^{(0)}$ that is a multiple of Δx?

   (f) (7 points) The logarithmic barrier subproblem is

$$\begin{array}{llrcrcrcl} \min & 10x_1 & + & x_2 & - & \mu \sum_{i=1}^{3} \ln(x_i) & & \\ \text{s.t.} & x_1 & - & x_2 & + & 2x_3 & = & 3 \\ & & & x_2 & - & x_3 & = & 2 \\ & & & & x_1, x_2, x_3 & & \geq & 0 \end{array}$$

   What equations would you solve to verify that $\bar{x} = (1, 6, 4)$ solves the barrier subproblem for $\mu = \frac{108}{7}$? (Note: you do not need to actually solve the equations.)

(a) $x^{(0)} > 0$ ✓ $\quad x_1^{(0)} - x_2^{(0)} + 2x_3^{(0)} = 3$ ✓ $\quad x_2^{(0)} - x_3^{(0)} = 2$ ✓

(b)
$$\begin{array}{ll} \min & 40x_1 + 3x_2 \\ \text{s.t.} & 4x_1 - 3x_2 + 2x_3 = 3 \\ & 3x_2 - x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(c) $c^T \Delta x = 7.669(-10+1) = -9(7.669) < 0$ ✓

$A\Delta x = 7.669 \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

(d) Need $x^{(0)} + \lambda \Delta x \geq 0$. So need $4 - \lambda(7.669) \geq 0$, so $\lambda_{max} = \frac{4}{7.669}$.

(e) Dimension of feasible region $\leq$ #vars − # constraints $= 1$.

(f) $x_i z_i = \mu$, so $z_1 = \frac{108}{7}$, $z_2 = \frac{108}{42} = \frac{18}{7}$, $z_3 = \frac{108}{28} = \frac{27}{7}$. Need $y_1 + z_1 = 10$, $-y_1 + y_2 + z_2 = 1$, $2y_1 - y_2 + z_3 = 0$ simultaneously.

2. (10 points) Express the following fixed charge problem as an equivalent integer programming problem:

$$\begin{array}{lrcrcl}
\min & c_1(x_1) & + & c_2(x_2) & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 10 \\
 & 2x_1 & + & x_2 & \geq & 10 \\
 & 0 & \leq & x_1, x_2 & \leq & 10
\end{array}$$

where

$$c_1(x_1) = \begin{cases} 0 & \text{if } x_1 = 0 \\ 2 + 2x_1 & \text{otherwise} \end{cases}$$

$$c_2(x_2) = \begin{cases} 0 & \text{if } x_2 = 0 \\ 1 + 3x_2 & \text{otherwise} \end{cases}$$

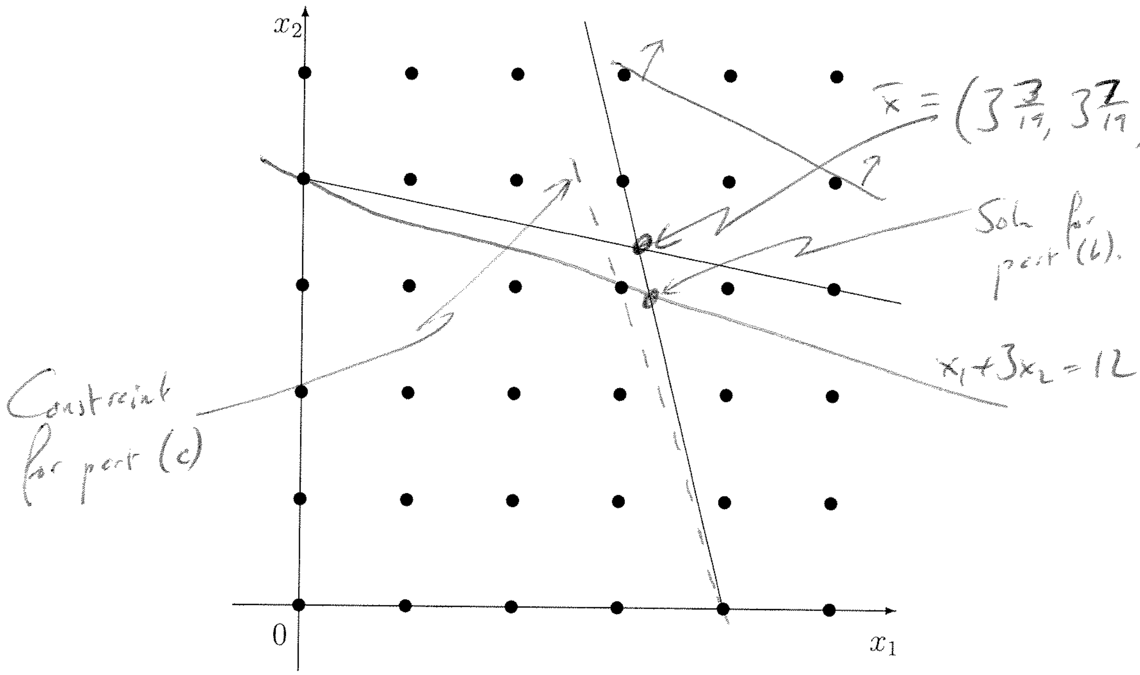
$$\begin{array}{lrcl}
\min & 2x_1 + 3x_2 + 2y_1 + y_2 & & \\
\text{s.t.} & x_1 + 2x_2 & \geq & 10 \\
 & 2x_1 + x_2 & \geq & 10 \\
 & x_1 - 10y_1 & \leq & 0 \\
 & x_2 - 10y_2 & \leq & 0 \\
 & x_1, x_2 \geq 0 & & \\
 & y_1, y_2 \text{ binary.} & &
\end{array}$$

3. (20 points)

   Consider the integer program

$$
\begin{array}{llcll}
\max & x_1 & + & 2x_2 & \\
\text{s.t.} & x_1 & + & 5x_2 & \leq 20 \\
 & 4x_1 & + & x_2 & \leq 16 \\
 & & & x_1, x_2 & \geq 0, \text{ integer}
\end{array}
\qquad (IP)
$$

The optimal solution to the linear programming relaxation of this problem is $\bar{x} = (3\frac{\overset{3}{\cancel{7}}}{19}, 3\frac{\overset{7}{\cancel{3}}}{19})$. The problem is illustrated here:



(a) (6 points) Show that the constraint $x_1 + 3x_2 \leq 12$ is a valid constraint (that is, the constraint is satisfied by all feasible solutions to (IP)).

(b) (6 points) Solve graphically the LP relaxation with the extra constraint $x_1 + 3x_2 \leq 12$.

(c) (8 points) Find a valid constraint that is violated by the point you found in part (b). Add the constraint to the LP relaxation and graphically solve it again.

(a) Constraint goes through $(0,4)$ and $(3,3)$.

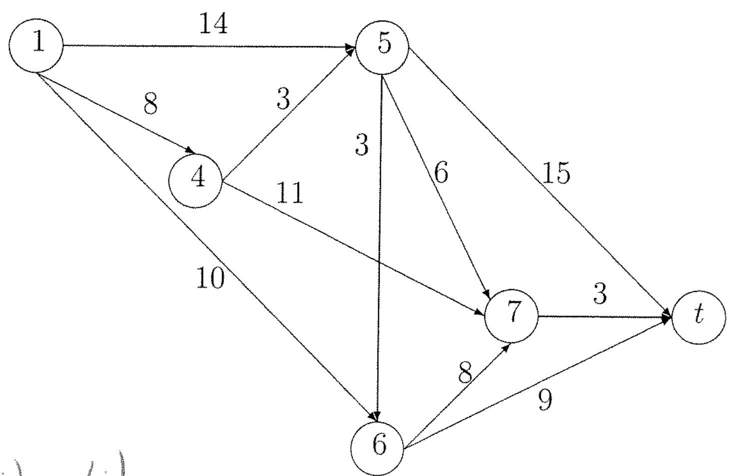
(b) Have $\left.\begin{array}{l} x_1+3x_2=12 \\ 4x_1+x_2=16 \end{array}\right\} \Rightarrow x_1 = \frac{36}{11} = 3\frac{3}{11}, \quad x_2 = 2\frac{10}{11}$.

(c) Add constraint through $(4,0)$ and $(3,3)$, namely $3x_1 + x_2 \leq 12$.

Soln is then $x=(3,3)$, so this solves (IP).

4. (15 points)

   Use Dijkstra's algorithm to find the shortest path from vertex 1 to vertex t in the following graph, where the edge lengths are given on the graph:



$v(i)$ = distance of shortest path to i

$p(i)$ = predecessor.

$v(i), p(i)$

Iteration \ Node	1	4	5	6	7	t
1	$v(1)=0$ fix	8, 1	14, 1	10, 1	—	—
2		fix	11, 4		19, 4	—
3				fix	18, 6	19, 6
4			fix		17, 5	
5					fix	
6						fix

Shortest path has length 19.

Backtracking: $1 \longrightarrow 6 \longrightarrow t$.

5. (15 points)

Burgess Fuel is planning its purchases of heating oil for the next four months. The forecasted demands and purchase costs per gallon are contained in the table:

	month 1	month 2	month 3
demand	7000	8000	3000
price per gallon	\$2	\$3	\$4

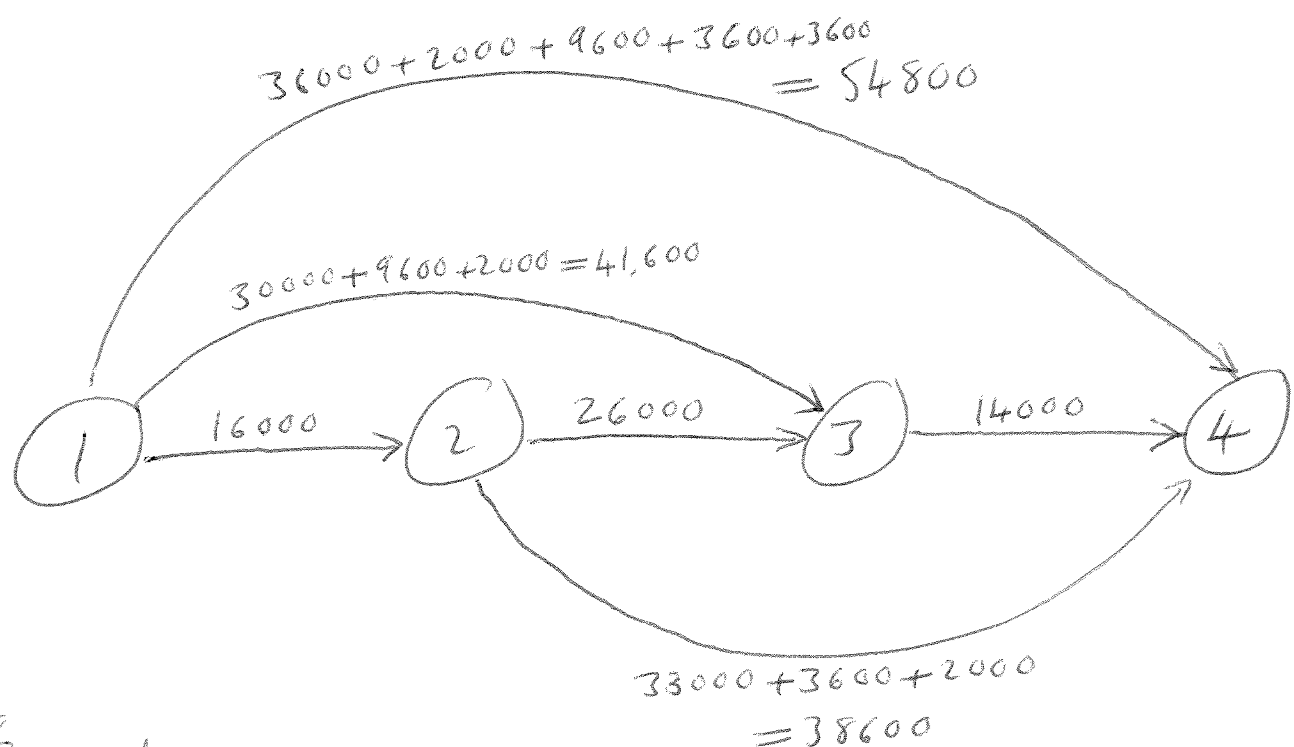
If they purchase a gallon of oil for use in a later month, they pay an inventory cost of \$1.20 per gallon per month in storage. Each order incurs a fixed delivery cost of \$2000, regardless of the amount ordered. Burgess Fuel currently has no fuel on hand.

~~(a) (5 points) Explain why~~ This problem can be approached by dynamic programming, with states $k = 1, \ldots, 4$ representing the reaching of month k with all earlier demand fulfilled and no inventory on hand.

~~(b) (5 points)~~ Sketch the digraph corresponding to the dynamic program ~~structure of part (a)~~. Include costs on all arcs.

~~(c) (5 points)~~ Explain why the feasible production plans correspond exactly to the paths from node $k = 1$ to node $k = 4$ in your digraph.

~~(a)~~



(c) Arc from node i to node j corresponds to purchasing oil for months i, ..., j-1.
Any path from 1 to 4 then corresponds to purchasing oil for all three months.