A Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

Second Exam, Friday, November 11, 2005.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts one hundred and ten minutes.

SOLUTIONS.

Q1		/ 40
Q2		/ 10
Q3		/ 15
Q4		/ 35
Total		/ 100
Grade		

1. (40 points)

   Consider the linear program:

$$
\begin{array}{lrcrcrcrcrcl}
\max & 6x_1 & + & x_2 & + & 21x_3 & - & 54x_4 & - & 8x_5 & & \\
\text{s.t.} & 2x_1 & & & + & 5x_3 & & & + & 7x_5 & = & 70 \\
 & & & 3x_2 & + & 3x_3 & - & 9x_4 & + & x_5 & = & 1 \\
 & & & & & & & & x_1,\ldots, & x_5 & \geq & 0
\end{array}
\qquad (P)
$$

   (a) (20 points) Compute the dual solution corresponding to each of the following basic sets in (P):

      i. $\{x_1, x_2\}$

      ii. $\{x_1, x_3\}$

      iii. $\{x_2, x_3\}$

      iv. $\{x_4, x_5\}$

   (b) The dual of a linear program of the form $\max\{c^T x : Ax = b, x \geq 0\}$ is of the form $\min\{b^T y : A^T y \geq c, y \text{ free}\}$.

      i. (6 points) Show that exactly two of the dual solutions you calculated in part (a) are feasible in the dual problem to (P).

      ii. (7 points) Use the information you determined in part 1(b)i to find the optimal solutions to (P) and its dual. (You may assume that any basic set not considered in part (a) is not optimal.)

      iii. (7 points) In this part, we are going to modify the right hand side of (P). This does not change the dual constraints, so it does not change the dual solution corresponding to a particular basis. Specifically, let the right hand side of the first constraint be 45 and that of the second constraint be 27. How does your answer to part 1(b)ii change? (Again, you may assume that any basic set not considered in part (a) is not optimal.)

(a) (i) $\left.\begin{array}{l} 2y_1 = 6 \\ 3y_2 = 1 \end{array}\right\} \Rightarrow y = \begin{bmatrix} 3 \\ 1/3 \end{bmatrix}$

(ii) $\left.\begin{array}{l} 2y_1 = 6 \\ 5y_1 + 3y_2 = 21 \end{array}\right\} \Rightarrow y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(iii) $\left.\begin{array}{l} 3y_2 = 1 \\ 5y_1 + 3y_2 = 21 \end{array}\right\} \Rightarrow y = \begin{bmatrix} 4 \\ 1/3 \end{bmatrix}$

(iv) $\left.\begin{array}{l} -9y_2 = -54 \\ 7y_1 + y_2 = -8 \end{array}\right\} \Rightarrow y = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

(intentionally left blank)

(b) Dual:

$$\begin{aligned} \max\quad & 70y_1 + y_2 \\ \text{s.t.}\quad & 2y_1 \geq 6 \\ & 3y_2 \geq 1 \\ & 5y_1 + 3y_2 \geq 21 \\ & -9y_2 \geq -54 \\ & 7y_1 + y_2 \geq -8 \end{aligned}$$

(i) $y = \begin{bmatrix} 3 \\ 1/3 \end{bmatrix}$: violates $5y_1 + 3y_2 \geq 21$

$y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$: feasible

$y = \begin{bmatrix} 4 \\ 1/3 \end{bmatrix}$: feasible

$y = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$: violates $2y_1 \geq 6$ and $5y_1 + 3y_2 \geq 21$.

(ii) Only possibilities are $\{x_1, x_3\}$ and $\{x_2, x_3\}$.

$\{x_1, x_3\}$: Need $\left.\begin{aligned} 2x_1 + 5x_3 &= 70 \\ 3x_3 &= 1 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{bmatrix} 34\frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$

$\{x_2, x_3\}$: Need $\left.\begin{aligned} 5x_3 &= 70 \\ 3x_2 + 3x_3 &= 1 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -13\frac{2}{3} \\ 14 \end{bmatrix}$ Infeasible.

So $\{x_1, x_3\}$ is the optimal primal basis,

and $y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is the optimal dual solution.

(intentionally left blank)

(iii) Look again at $\{x_1, x_3\}$ and $\{x_2, x_3\}$:

$$\{x_1, x_3\}: \quad \left.\begin{aligned} 2x_1 + 5x_3 &= 45 \\ 3x_3 &= 27 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix}$$

$$\{x_2, x_3\}: \quad \left.\begin{aligned} 5x_3 &= 45 \\ 3x_2 + 3x_3 &= 27 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix}.$$

So both bases are feasible.

Optimal primal soln: $x = (0, 0, 9, 0, 0)$.

Both $y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 4 \\ 1/3 \end{bmatrix}$ (and convex combos of them) are dual optimal.

2. (10 points)

   A linear program in standard form has optimal tableau:

$$M = \begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline 6 & 0 & 0 & 0 & 3 & 2 & 1 \\ \hline \frac{1}{2} & 1 & 0 & 0 & 2 & 4 & 1 \\ \frac{1}{2} & 0 & 0 & 1 & -3 & 2 & -1 \\ \frac{1}{2} & 0 & 1 & 0 & 1 & -1 & -1 \end{array}$$

   An additional constraint is now added:

$$x_4 + x_5 + x_6 \geq \frac{1}{2}.$$

   Use the dual simplex algorithm to show that the value, of ~~every variable~~ ~~is~~ x_1, x_2, and x_3 are integral in the optimal solution to the modified problem.

Add constraint $-x_4 - x_5 - x_6 + x_7 = -\frac{1}{2}$:

6	0	0	0	3	2	1	0
$\frac{1}{2}$	1	0	0	2	4	1	0
$\frac{1}{2}$	0	0	1	-3	2	-1	0
$\frac{1}{2}$	0	1	0	1	-1	-1	0
$\rightarrow$ $-\frac{1}{2}$	0	0	0	-1	-1	-1	1
				$3/1$	$2/1$	$1/1$	

Pivot:

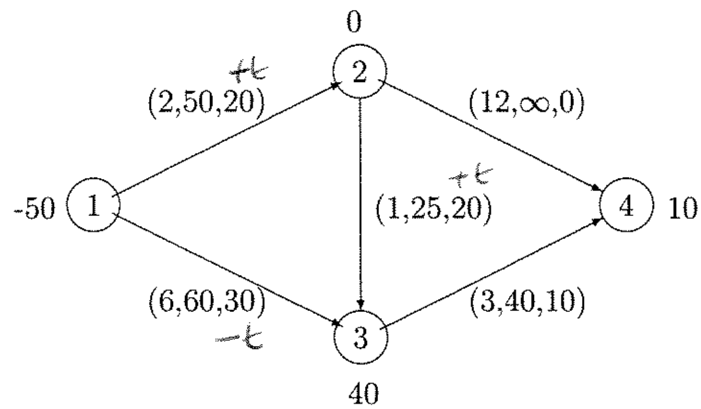
$5\frac{1}{2}$	0	0	0	2	1	0	1
0	1	0	0	1	3	0	1
1	0	0	1	-2	3	0	-1
1	0	1	0	2	0	0	-1
$\frac{1}{2}$	0	0	0	1	1	1	-1

So $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_6 = \frac{1}{2}$

$x_4 = x_5 = x_7 = 0$

3. (15 points)

   The following digraph represents a partially solved minimum cost flow problem with labels on nodes indicating net demand and those on arcs showing unit cost, capacity, and current flow.



   The given feasible solution is not a basic feasible solution. Find a basic feasible solution that is at least as good as the given solution.

   Is your basic feasible solution optimal?

Have cycle $1 \to 2 \to 3 \to 1$. Adjust flow on cycle

Change in value $= t(2+1-6) = -3t$.

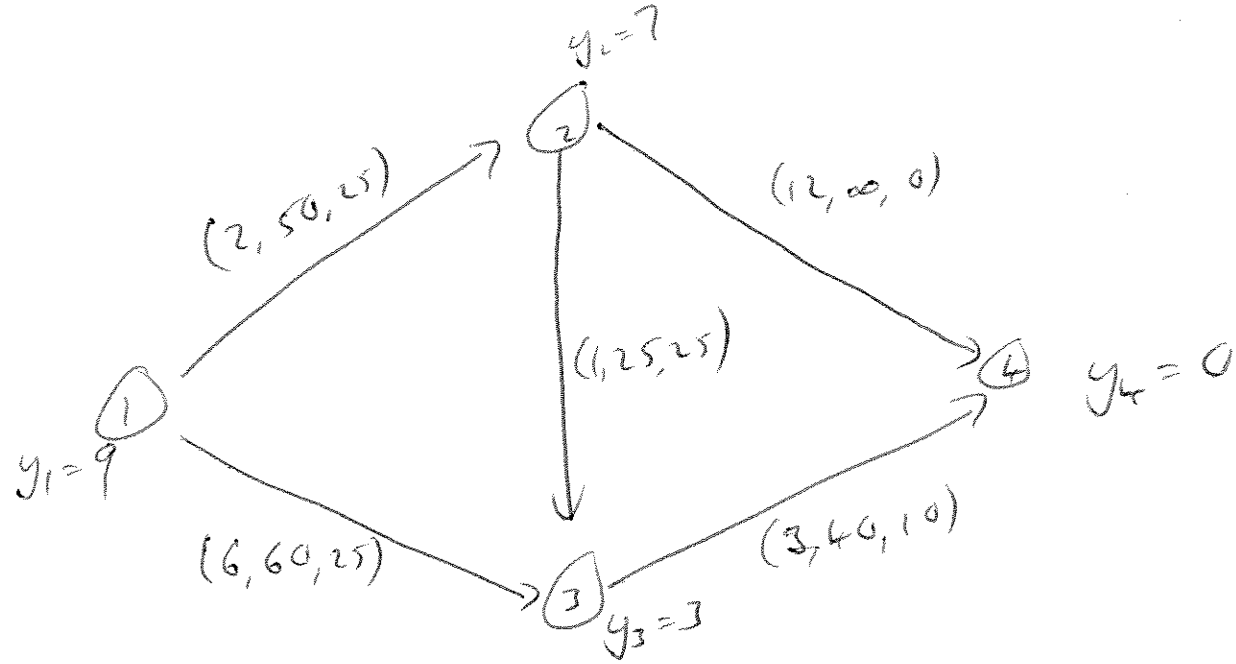
So want $t > 0$.

Largest possible value is $t = 5$, when $x_{23} = 25$, its upper bound.

So x_{23} becomes nonbasic at its upper bound.

(intentionally left blank)

Basic feasible solution:



For basic arcs $(1,2)$, $(1,3)$, $(3,4)$, have $y_i - y_j = c_{ij}$.

So if $y_4 = 0$, get $y_3 = 3$, $y_1 = 9$, $y_2 = 7$

For nonbasics at lower bound:

optimal if $c_{ij} \geq y_i - y_j$. $(2,4)$: $y_i - y_j = 7 - 0 = 0 < 12 = c_{24}$ ✓

For nonbasics at upper bound:

optimal if $c_{ij} \leq y_i - y_j$ $(2,3)$: $y_2 - y_3 = 7 - 3 = 4 > 1 = c_{23}$ ✓.

So YES, it is optimal.

4. (35 points)

A transportation problem with three supply nodes and four demand nodes has tableau

					u_i
	6	3^{10}	1^{20}	8	0
	2^{20}	4	4	7	-2
	3^{20}	2^{10}	3	5^{20}	-1
v_j	4	3	1	6	

The large numbers give the costs of the arcs and the superscripts indicate a flow. This flow is a basic feasible solution.

(a) (7 points) Show that the reduced costs for this basic feasible solution are as follows:

2	0^{10}	0^{20}	2
0^{20}	3	5	3
0^{20}	0^{10}	3	0^{20}

(b) (7 points) Currently, the cost of the arc from supply node 1 to demand node 4 is $c_{14} = 8$. What is the smallest value of c_{14} for which the given basic feasible solution is optimal?

(c) (7 points) Now set $c_{14} = 4$. Find the optimal solution. (Hint: only one pivot is needed.)

(d) For this part, reset $c_{14} = 8$.

i. (7 points) Assume the supply at node 2 and the demand at node 3 are each increased by 5 units. **Use sensitivity analysis** to find the change in the optimal value.

ii. (7 points) Now consider increasing the supply at node 2 and the demand at node 3 by t units. What is the maximum value for t for which the rate you found in part 4(d)i still holds? Would you expect the rate to be larger or smaller for larger values of t?

(a) See above for u_i, v_j calculation ($u_i + v_j = c_{ij}$ for basic arcs)

For nonbasics, reduced costs $= c_{ij} - u_i - v_j$

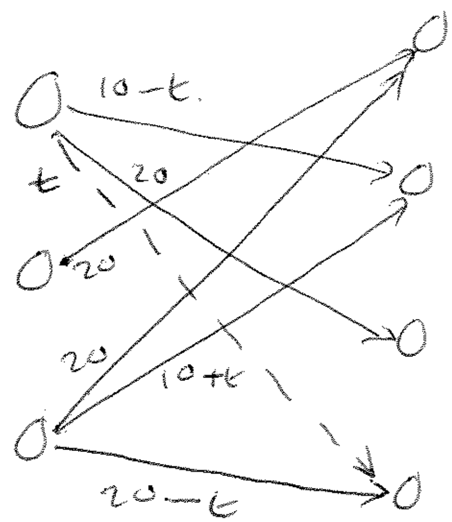
It can be verified that this gives the stated table.

(b) If c_{14} is decreased by more than 2 then its reduced cost will be negative.

So smallest value for this basis to be in optimal form is $8 - 2 = \boxed{6}$.

(intentionally left blank)

(c) If $c_{14} = 4$: x_{14} enters the basis



Maximum value for t is 10.

x_{12} leaves the basis.

				u_i
6	3	1^{20}	$\cancel{8}4^{10}$	0
2^{20}	4	4	7	0
3^{20}	2^{20}	3	5^{10}	1
v_j: 2	1	1	4	

Reduced costs:

4	2	0^{20}	0^{10}
0^{20}	3	3	3
0^{20}	0^{20}	1	0^{10}

Optimal

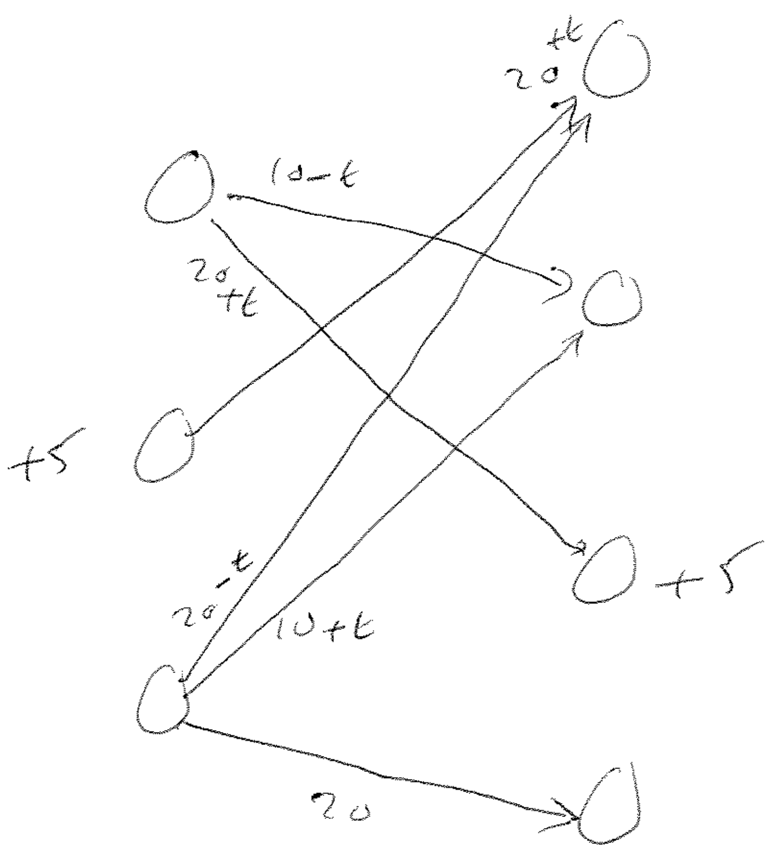
(intentionally left blank)

(d) (i) Change in value

$$= 5u_2 + 5v_3 = 5(-2) + 5(1) = -5.$$

Thus, the optimal value <u>decreases</u> by 5.

(ii) To be able to keep the same optimal basis, need to change only the basic variables:



Adjust flow along the path from Supply 2 to Demand 3.

Largest possible value is $\boxed{t=10}$ when x_{12} goes to zero

Old rate = −1

New rate: > −1

since constraints become harder to satisfy.