1. (a) Hessian of constraint is $\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$, so $g(x) = 4x_1^2 + x_2^2 - 400$ is convex, so feasible region is convex.

$f(x) = -x_1^2 - (x_2 - 9)^2$, so $D^2 f = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$,

and $D^2(-f) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. So $-f$ is convex, so f is concave

SLATER CQ holds: $x = (0,0)$ is strictly feasible.

(b) $Df = \begin{bmatrix} -2x_1 \\ -2(x_2 - 9) \end{bmatrix}$ $\quad Dg = \begin{bmatrix} 8x_1 \\ 2x_2 \end{bmatrix}$

At $x^1 = (8, 12)$:

$Df = \begin{bmatrix} -16 \\ -6 \end{bmatrix}$ $\quad Dg = \begin{bmatrix} 64 \\ 24 \end{bmatrix}$

$u^1 = \frac{1}{4}$ gives $Df(x^1) + u^1 Dg(x^1) = 0$.

Also, $g(x^1) = 0$, so $u^1 g(x^1) = 0$.

At $x^2 = (0, -20)$:

$Df = \begin{bmatrix} 0 \\ +58 \end{bmatrix}$, $\quad Dg = \begin{bmatrix} 0 \\ -40 \end{bmatrix}$

$u^2 = \frac{58}{40} = \frac{29}{20} = 1.45$ gives $Df(x^2) + u^2 Dg(x^2) = 0$

Also, $g(x^2) = 0$, so $u^2 g(x^2) = 0$.

(c) x^1: $D^2 f(x^1) + \mu^1 D^2 g(x^1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\cancel{\tfrac{7}{4}}\tfrac{3}{2} \end{bmatrix}. =: D^2 L(x^1)$$

Interested in directions d satisfying $d^T Dg(x^1) = 0$.

So $\begin{bmatrix} 64 \\ 24 \end{bmatrix}^T \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0$, so $d = \alpha \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ for any α.

$d^T D^2 L(x^1) d = \alpha^2(-112) < 0$.

So x^1 is <u>not</u> a local minimizer.

x^2: $D^2 L(x^2) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + 1.45 \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9.6 & 0 \\ 0 & 0.9 \end{bmatrix}$

Want $d^T Dg(x^2) = 0$, so $\begin{bmatrix} 0 \\ -40 \end{bmatrix}^T d = 0$, so $d = \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for any β.

Then $d^T D^2 L(x^2) d = 9.6\beta^2 > 0$ for $\beta \neq 0$.

So x^2 <u>is</u> a local minimizer.

(d) Need

$$-2x_1 + 8ux_1 = 0 \quad (1)$$
$$-2(x_2-9) + 2ux_2 = 0 \quad (2)$$

(1) $\Rightarrow$ either $x_1 = 0$ or $u = \frac{1}{4}$.

Also need $u(4x_1^2 + x_2^2 - 400) = 0$ (3)

If $x_1 = 0$: From (3), either $u = 0$ or $x_2 = \pm 20$.

Already looked at $x^2 = (0, -20)$.

Check $(0, 20)$: From (2), need $u = \frac{22}{40} = 0.55$.

So get another KKT point: $x^3 = (0, 20)$.

~~If u = 0~~

If $u = 0$, and $x_1 = 0$:

From (2), $x_2 = 9$.

Get another KKT point: $x^4 = (0, 9)$.

If $u_1 = \frac{1}{4}$:

From (2), need $18 = \frac{3}{2} x_2$, so $x_2 = 12$.

From (3), need $4x_1^2 = 256$, or $x_1^2 = 64$.

Get another KKT point: $x^5 = (-8, 12)$.

(already seen $x^1 = (8, 12)$).

(e) Optimal soln must be a KKT point, since CQ holds.

By Weierstrass, there exists an optimal soln.

Thus, it suffices to compare x^1, x^2, x^3, x^4, x^5:

$$f(x^1) = -64 - 9 = -73$$

$$f(x^2) = 0 - 841 = -841$$

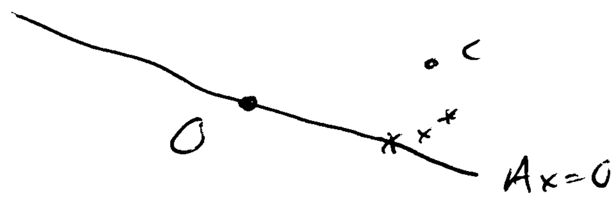
$$f(x^3) = 0 - 121 = -121$$

$$f(x^4) = 0 + 0 = 0$$

$$f(x^5) = -64 - 9 = -73$$

So $x^2 = (0, -20)$ is optimal

2. (a) Find the closest point on the plane $Ax=0$ to the point c.



(b) Let y satisfy $AA^Ty = 0$.
So A^Ty is in the nullspace of A and also in the row space of A.

So $A^Ty = 0$.

Since A has full row rank, must have $y=0$.

(c) $\nabla f = c - x$, $\nabla g = A^T$

ith column = ith row of A
= gradient of linear constraint $a_i^T x = 0$, with a_i = ith row of A.

Need $c - x - A^T v = 0$

Let $\bar{v} = (AA^T)^{-1}Ac$, so $\bar{x} = c - A^T(AA^T)^{-1}Ac$

This is a KKT provided it is feasible.

Check: $A\bar{x} = Ac - AA^T(AA^T)^{-1}Ac = 0$ ✓

(d) $f(x) = \frac{1}{2}\|c - x\|_2^2$ is strictly convex, since $\nabla^2 f = I$.

Constraints are linear.

Thus, any KKT point is the global minimizer.

(e) $f(\bar{x}) = \frac{1}{2}\|c - \bar{x}\|_2^2 = \frac{1}{2}\|A^T(AA^T)^{-1}Ac\|_2^2$

$$= \frac{1}{2} c^T A^T (AA^T)^{-1} A A^T (AA^T)^{-1} A c$$

$$= \frac{1}{2} c^T A^T (AA^T)^{-1} A c$$

To get $f(x) = 0$, need $x = c$.

For this to be feasible, need $Ac = 0$.

So $\cancel{f}$ optimal value $= 0 \Rightarrow Ac = 0$.

Conversely, $Ac = 0 \Rightarrow x = c$ is feasible $\Rightarrow$ opt value is zero.

3.

$$
\begin{aligned}
\theta(u) &= \inf_{x \in K} \; c^T x + u^T(b - Ax) \\
&= b^T u + \inf_{x \in K} \; (c - A^T u)^T x \\
&= \begin{cases} b^T u & \text{if } c - A^T u \in K^*, \text{ since then } (c - A^T u)^T x \geq 0 \ \forall x \in K, \text{ with equality if } x = 0 \\ -\infty & \text{if } (c - A^T u) \notin K^*, \text{ since then } \exists \bar{x} \in K \text{ with } (c - A^T u)^T \bar{x} < 0, \text{ and taking } x = \alpha \bar{x} \text{ lets us drive } (c - A^T u)^T x \to -\infty \text{ as } \alpha \to +\infty. \end{cases}
\end{aligned}
$$

3(b) ~~Both~~

Only primal feasible point is $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, with value 1

Dual problem is

$$\begin{aligned} \max \quad & u_2 \\ \text{s.t.} \quad & \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - u_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - u_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{c - A^T u = \begin{bmatrix} -u_1 - u_2 & +u_1 \\ u_1 & 1 - u_1 \end{bmatrix}} \text{ is sym, psd.} \end{aligned}$$

That is,

$$\begin{aligned} \max \quad & u_2 \\ \text{s.t.} \quad & u_1 + u_2 \leq 0 \\ & 1 - u_1 \geq 0 \\ & (u_1 + u_2)(1 - u_1) + u_1^2 \leq 0 \end{aligned}$$

Ie,

$$\begin{aligned} \max \quad & u_2 \\ \text{s.t.} \quad & u_1 + u_2 \leq 0 \\ & 1 - u_1 \geq 0 \\ & u_1 + u_2 - u_1 u_2 \leq 0 \end{aligned}$$

~~Now, $u_2 > 0$ ... $u_1 \leq -u_2 \leq 0$~~

~~Also $u_1 + u_2 \leq 0$~~

- Take $u_2 = 1 - \varepsilon$, $u_1 = \frac{u_2}{1 - u_2} = -\frac{(1-\varepsilon)}{\varepsilon}$, $\varepsilon > 0$.

  Then $c - A^T u = \begin{bmatrix} \varepsilon - 1 + \frac{(1-\varepsilon)}{\varepsilon} & -\frac{1-\varepsilon}{\varepsilon} \\ -\frac{1-\varepsilon}{\varepsilon} & 1 + \frac{(1-\varepsilon)}{\varepsilon} \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} \varepsilon^2 - 2\varepsilon + 1 & -1 + \varepsilon \\ -1 + \varepsilon & 1 \end{bmatrix}$

  Determinant is $\varepsilon^2 - 2\varepsilon + 1 - (\varepsilon - 1)^2 = 0$, so feasible.

- Take $u_2 = 1$. Need $(1 - u_1)(-1 - u_1) - u_1^2 \geq 0$, ie, $-1 \geq 0$.

  So $\sup \theta(u) = 1$, but this is not achieved.

4. Let $v_i(t)$, $i=1,\dots,4$ satisfy

$$v_1(t) \cancel{+} - v_2(t) + v_3(t) - v_4(t) = u(t),$$

$$0 \le v_i(t) \quad \forall i$$

$$v_1(t) \le 1, \quad v_2(t) < 1$$

Then $f(u(t)) = v_1(t) + v_2(t) + 2v_3(t) + 2v_4(t)$,

provided
$$v_1(t) \cdot v_2(t) = 0$$
$$v_1(t) \cdot v_4(t) = 0$$
$$v_2(t) \cdot v_3(t) = 0$$
$$v_3(t) \cdot v_4(t) = 0$$
$$v_3(t) > 0 \text{ only if } v_1(t) = 1$$
$$v_4(t) > 0 \text{ only if } v_2(t) = 1.$$

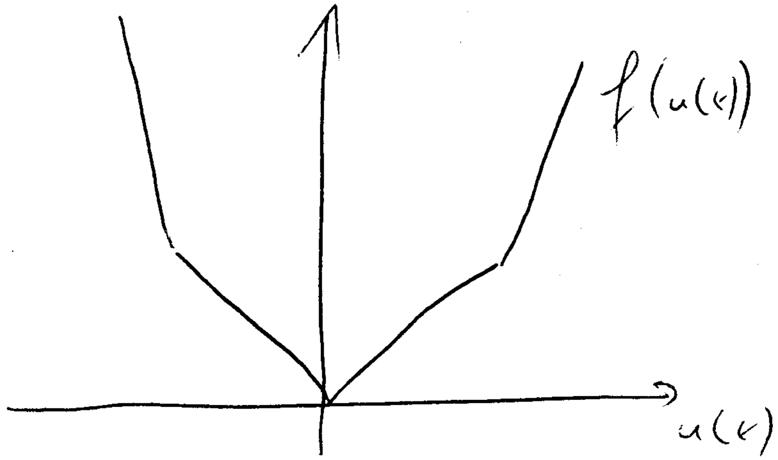
Note that minimizing $f(u(t))$ will automatically get these conditions satisfied.

So:
$$\min \sum_{t=1}^{T} \left(v_1(t) + v_2(t) + 2v_3(t) + 2v_4(t)\right)$$

s.t.
$$u(t) = v_1(t) - v_2(t) + v_3(t) - v_4(t), \quad t=1,\dots,T$$
$$x(t) = Ax(t-1) + bu(t), \quad t=1,\dots,T$$
$$x(T) = \bar{x}$$
$$x(0) = 0$$
$$v_i(t) \ge 0, \quad i=1,\dots,4, \quad t=1,\dots,T$$
$$v_1(t),\ v_2(t) \le 1, \quad t=1,\dots,T$$

Alternative formulation for ~~3(a)~~ 4:

$$f(u(t)) = \max\{u(t), -u(t), 2u(t)-1, -2u(t)-1\}$$



So:

$$\min \sum_{t=1}^{T} z(t)$$

$$\text{s.t.} \quad \left.\begin{aligned} z(t) &\geq u(t) \\ z(t) &\geq -u(t) \\ z(t) &\geq 2u(t)-1 \\ z(t) &\geq -2u(t)-1 \\ x(t) &= Ax(t-1)+bu(t) \end{aligned}\right\} \quad t=1,\dots,T$$

$$x(T) = \bar{x}$$

$$x(0) = 0$$

At the optimal, $z(t)$ will be pushed down to equal $f(u(t))$.

5. $u^1 = (1,1)$.

$$L(x,u^1) = x_1^2 + 4x_2^2 + 6x_3^2 + 3(x_1-4)^2 + 4(x_2+2)^2 + (x_3-5)^2 - 40 + 2 - x_1$$

$$\frac{\partial L}{\partial x_1} = 2x_1 + 6(x_1-4) - 1 = 8x_1 - 25 \Rightarrow x_1(u^1) = \frac{25}{8}$$

$$\frac{\partial L}{\partial x_2} = 8x_2 + 8(x_2+2) = 16x_2 + 16 \Rightarrow x_2(u^1) = -1$$

$$\frac{\partial L}{\partial x_3} = 12x_3 + 2(x_3-5) = 14x_3 - 10 \Rightarrow x_3(u^1) = \frac{5}{7}.$$

$$f(x(u^1)) = \left(\frac{25}{8}\right)^2 + 4(1)^2 + 6\left(\frac{5}{7}\right)^2 = \cancel{14.605}\ 16.827$$ (rounding up)

$$g_1(x(u^1)) = 3\left(\frac{7}{8}\right)^2 + 4(1)^2 + \left(\frac{30}{7}\right)^2 - 40 = \cancel{-12.225}\ -15.336$$

$$g_2(x(u^1)) = -\frac{25}{8} + 2 = -\frac{9}{8}$$

Thus, (DA1) is:

$$\begin{array}{ll} \max & z \\ \text{s.t.} & z \le \cancel{14.605}\ 16.827 - \cancel{12.225}\ 15.336\, u_1 - \frac{9}{8} u_2 \\ & u \ge 0 \end{array} \qquad (DA1)$$

Optimized at $u^2 = (0,0)$, with $z = \cancel{14.605}\ 16.827$

$L(x, u^2) = x_1^2 + 4x_2^2 + 6x_3^2,$

so $\quad x(u^2) = (0,0,0)$ and $f(x(u^2)) = 0, \quad g_1(x(u^2)) = 40,$
$g_2(x(u^2)) = -2.$

So (DA2) is

$$
\begin{aligned}
\max \quad & z \\
\text{s.t.} \quad & z \le \overset{16.827}{\cancel{14.605}} - \overset{15.336}{\cancel{}}\, u_1 - \tfrac{9}{8} u_2 \qquad \text{(DA2)} \\
& z \le -40 u_1 + 2 u_2 \\
& u \ge 0
\end{aligned}
$$

At the optimal soln, $\boxed{u_1 = 0.}$

Also, $\quad z = \overset{16.827}{\cancel{14.605}} - \frac{9}{8} u_2 = 2u_2$

Thus, $\quad u_2 = \dfrac{\overset{16.827}{\cancel{14.605}}}{25/8} = \underset{5.385}{\cancel{4.6736}}$ and $z = \underset{10.769}{\cancel{9.3472}}.$

So $\boxed{u^3 = (0, \overset{5.385}{\cancel{4.6736}}), \quad z^3 = \overset{10.769}{\cancel{9.3472}}}$

Upper bound:

(DAp) is a relaxation of the Lagrangian dual problem, so its optimal value is an upper bound.

So final upper bound is $\boxed{\cancel{\text{[illegible]}}}$ $\boxed{10.769}$

Note that $x(u^1)$ is primal feasible, so $f(x(u^1))$ also provides an upper bound. However, this is worse than the bound above.

Lower bound:

We have evaluated $\theta(u^1)$, $\theta(u^2)$, and u^1 and u^2 are both dual feasible. Thus, these values each provide a lower bound.

$$\theta(u^1) = \overset{16.827}{\cancel{\text{[illegible]}}} - \overset{15.336}{\cancel{\text{[illegible]}}} - \frac{9}{8} = \overset{0.366}{\cancel{\text{[illegible]}}}$$

$$\theta(u^2) = 0.$$

So best lower bound is $\boxed{\overset{0.366}{\cancel{\text{[illegible]}}}.}$

(Optimal value is ≈ 6.67)