A

Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

Third Exam, Friday, December 10, 2004.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of six questions, and lasts one hundred minutes.

Q1	/ 20
Q2	/ 20
Q3	/ 35
Q4	/ 25
Total	/ 100
Grade	

SOLUTION.

1. (20 points)

   A pharmaceutical manufacturer must supply 30 batches of its new medication in the next quarter, then 25, 10, and 35 in successive quarters. Each quarter in which the company makes product requires \$100,000 setup, plus \$3000 per batch produced. There is no limit on production capacity. Batches can be held in inventory, but the cost is a high \$5000 per batch per quarter. The company seeks a minimum total cost production plan.

   (a) (5 points)

   Explain why this problem can be approached by dynamic programming, with states $k = 1, \ldots, 5$ representing the reaching of quarter k with all earlier demand fulfilled and no inventory on hand.

   (b) (10 points)

   Sketch the digraph corresponding to the dynamic program structure of part (a). Include costs on all arcs.

   (c) (5 points)

   Explain why the feasible production plans correspond exactly to the paths from node $k = 1$ to node $k = 5$ in your digraph.

Rardin, 9.26 (a)-(c)

2. (20 points)

A **node packing** on a graph $G = (V, E)$ is a subset of the vertices so that no two vertices in the subset share an edge. If each vertex v has a weight w_v, the **weighted node packing** problem on a graph is to choose a node packing $P \subset V$ of maximum weight $\sum_{v \in P} w_v$. This can be formulated as an integer programming problem by introducing a binary variable x_v for each vertex $v \in V$ and then solving the problem

$$\begin{array}{lrcl} \min & -\sum_{v \in V} w_v x_v & & \\ \text{subject to} & x_u + x_v & \leq & 1 \text{ for each edge } (u,v) \in E \\ & x_v & & \text{binary } \forall v \in V \end{array} \qquad (WNP)$$

Let G be the complete graph on three vertices $V = \{1, 2, 3\}$, so G has undirected edges (1,2), (2,3), and (1,3). Let $w_1 = 5$, $w_2 = 4$, and $w_3 = 3$. Solving the linear programming relaxation of (WNP) for this problem gives an optimal tableau

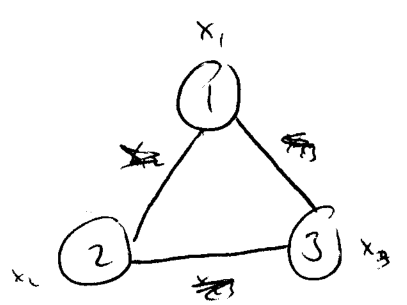
		x_1	x_2	x_3	s_1	s_2	s_3
	6	0	0	0	3	1	2
$M =$	0.5	1	0	0	0.5	-0.5	0.5
	0.5	0	1	0	0.5	0.5	-0.5
	0.5	0	0	1	-0.5	0.5	0.5

where s_1, s_2, and s_3 are the slack variables in (WNP).

(a) (10 points)
Why is the inequality $x_1 + x_2 + x_3 \leq 1$ a valid inequality that could be added to (WNP)?

(b) (10 points)
Add the inequality $x_1 + x_2 + x_3 \leq 1$ to M and reoptimize. (You should obtain an integral solution.)

(a)



If ~~every~~ any $x_i = 1$, then its neighbours j have $x_j = 0$.

So $x_1 = 1 \Rightarrow x_2 = x_3 = 0$ etc.

So $x_1 + x_2 + x_3 \leq 1$

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(b)

							s_4
C	0	0	0	3	1	2	0
$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
$\frac{1}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
1	1	1	1	0	0	0	1

Pivot to restore identity:

6	0	0	0	3	1	2	0
$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
$\frac{1}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
→ $-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
				$3/\frac{1}{2}$	$1/\frac{1}{2}$	$2/\frac{1}{2}$	

Dual simplex pivot:

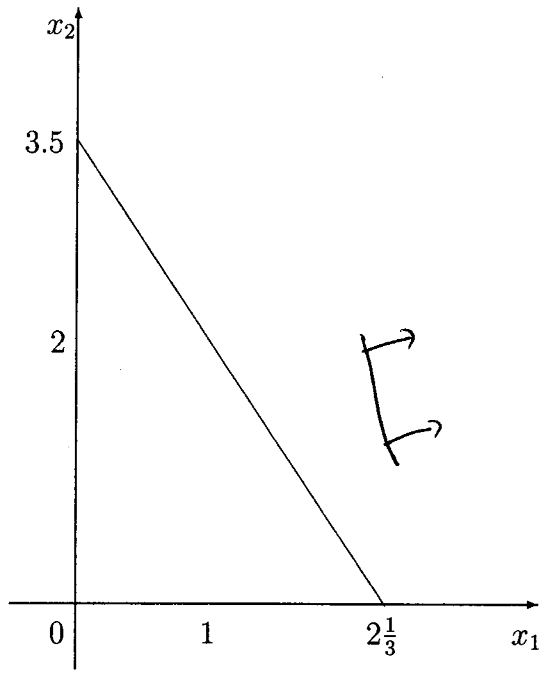
5	0	0	0	2	0	1	2
1	1	0	0	1	0	1	-1
0	0	1	0	0	0	-1	1
0	0	0	1	-1	0	0	1
1	0	0	0	1	1	1	-2

Optimal.
Soln: $x_1 = 1$, $x_2 = x_3 = 0$
$s_1 = 0$, $s_2 = 1$, $s_3 = 0$, $s_4 =$

3. Consider the integer programming problem

$$\begin{array}{llrcl} \max & 5x_1 & + & 3x_2 & \\ \text{subject to} & 3x_1 & + & 2x_2 & \leq 7 \\ & & & x_1, x_2 & \geq 0 \text{ and integer} \end{array} \qquad (IP)$$

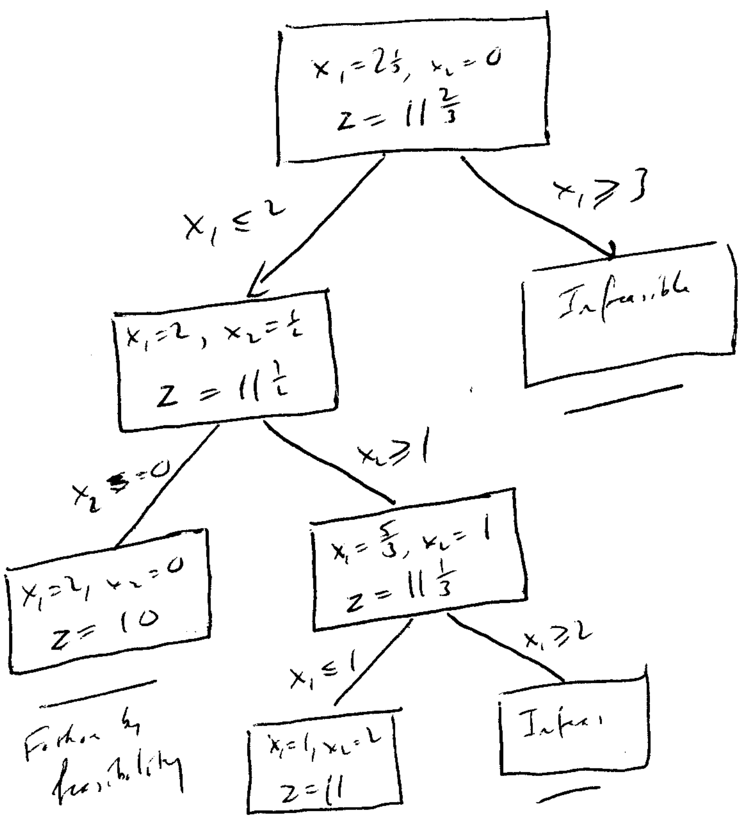
(a) (20 points)
Solve this integer program **using branch-and-bound**. It is acceptable to solve the LP relaxations graphically or by inspection; the problem is illustrated below.



$5(2\frac{1}{3}) = \cancel{10}\ 11\frac{2}{3}$

$3(3\frac{1}{2}) = 10\frac{1}{2}.$

Z = value = $5x_1 + 3x_2$



Opt value:
$x_1 = 1$, $x_2 = 2$,
value = 11

(b) (15 points)
Solve (IP) using **dynamic programming**. Be sure to state the recursion you use to get $f_1(s, x_1)$ in terms of $f_2(s)$, and to explain your stages and state variable(s).

s = remaining slack

At stage n, determine value of x_n to optimize allocation of slack over variables $n, \ldots, 2$

$$f_1(s, x_1) = 5x_1 + f_2(s - 3x_1)$$

Want to find $f_1(7)$

$$f_2(s) = \begin{cases} 0 & \text{if } s = 0 \text{ or } 1, \text{ with } x_2 = 0 \\ 3 & \text{if } s = 2 \text{ or } 3, \text{ with } x_2 = 1 \\ 6 & \text{if } s = 4 \text{ or } 5, \text{ with } x_2 = 2 \\ 9 & \text{if } s = 6 \text{ or } 7, \text{ with } x_2 = 3 \end{cases}$$

$f_1(7, x_1)$:

s	$f_1(7,x_1) = 5x_1 + f_2(7-3x_1)$ $x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$f_1(s)$	x_1^*
7	$0+9 = 9$	$5+6 = 11$	$10+0 = 10$	11	1

Thus, $x_1^* = 1$, $x_2^* = 2$, value = 11.

4. (25 points)

The linear programming problem

$$\begin{array}{llcl}
\min & -2x_1 - x_2 & & \\
\text{subject to} & 2x_1 + x_2 + x_3 & = & 8 \\
& -x_1 + 2x_2 + x_4 & = & 6 \\
& x_i & \geq & 0 \quad \text{for } i = 1, \ldots, 4
\end{array} \qquad (P)$$

has dual

$$\begin{array}{llcl}
\max & 8y_1 + 6y_2 & & \\
\text{subject to} & 2y_1 - y_2 & \leq & -2 \\
& y_1 + 2y_2 & \leq & -1 \\
& y_1 & \leq & 0 \\
& y_2 & \leq & 0
\end{array} \qquad (D).$$

(a) (3 points)

Let $\bar{x} = (3, 2, 0, 5)$ and $\bar{y} = (-1, 0)$. Show that this is an optimal solution to the primal-dual pair (P) and (D).

(b) (3 points)

Is $\bar{x}$ a basic feasible solution? Justify your answer.

(c) (3 points)

Find two optimal basic feasible solutions to (P), graphically. Label $\bar{x}$ on your graph.

(d) (3 points)

Show that (D) is degenerate, graphically. Label $\bar{y}$ on your graph.

(e) (3 points)

Give an integral feasible interior point for (P).

(f) (10 points)

Let $\tilde{x} = (3 - \epsilon, 2 - \epsilon, 3\epsilon, 5 + \epsilon)$ and $\tilde{y} = (-1 - \epsilon, -\epsilon)$, where $0 < \epsilon < 0.1$ Let $\tilde{z}$ denote the corresponding dual slacks. Show that all complementary slackness terms $\tilde{x}_i\tilde{z}_i$ lie between 2ϵ and 6ϵ. Hence argue that $\tilde{x}$ is close to being an optimal solution to the log barrier problem

$$\begin{array}{llcl}
\min & -2x_1 - x_2 - 4\epsilon \sum_{i=1}^{4} \ln(x_i) & & \\
\text{subject to} & 2x_1 + x_2 + x_3 & = & 8 \\
& -x_1 + 2x_2 + x_4 & = & 6 \\
& x_i & \geq & 0 \quad \text{for } i = 1, \ldots, 4
\end{array} \qquad (LBP)$$

(Hint: For example, you could argue that $\tilde{x}$ almost satisfies the optimality conditions for (LBP).)

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(a) x is primal feasible ✓

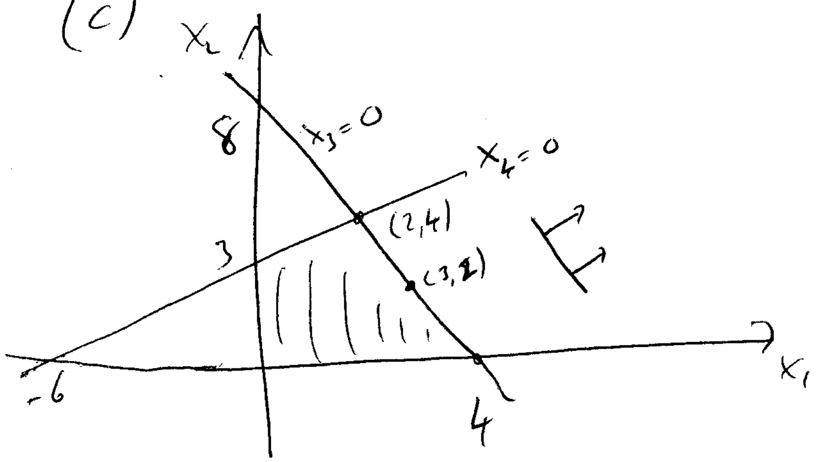
y is dual feasible ✓

$c^T x = -8$, $b^T y = -8$ ✓

(b) A BFS would have at most two positive components.

so $\bar{x}$ is $\boxed{\text{NOT A BFS}}$

(c)

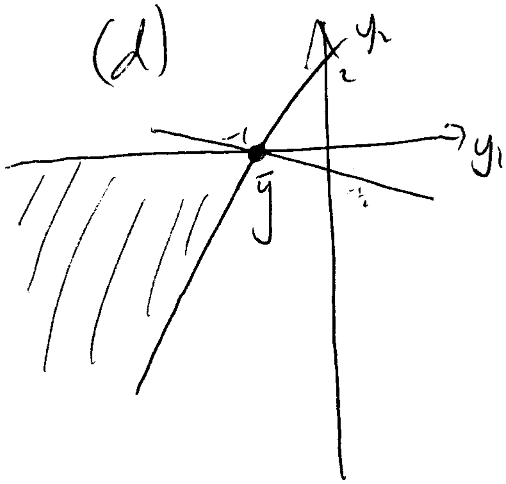


Optimal bfs:

$x = (2, 4, 0, 0)$

$x = (4, 0, 0, 10)$

(d)



Three constraints intersect at $\bar{y}$.

(e) Eg: $x = (1, 12, 5, 5)$

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(d) $\tilde{z} = c - A^T\tilde{y} = \begin{pmatrix} 0+\varepsilon \\ 0+3\varepsilon \\ 1+\varepsilon \\ 0+\varepsilon \end{pmatrix} \qquad \tilde{x} = \begin{pmatrix} 3-\varepsilon \\ 2-\varepsilon \\ 3\varepsilon \\ 5+\varepsilon \end{pmatrix}$

$\tilde{Z}\tilde{X}e = \begin{bmatrix} 3\varepsilon - \varepsilon^2 \\ 6\varepsilon - 3\varepsilon^2 \\ 3\varepsilon + 3\varepsilon^2 \\ 5\varepsilon + \varepsilon^2 \end{bmatrix}$, as required.

Optimality conditions for (LBP):

$Ax = b$

$A^Ty + z = c$

$x_i z_i = 4\varepsilon$

All violations are $\leq 2\varepsilon$.