B Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

First Exam, Friday, September 19, 2003.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of six questions, and lasts one hundred minutes.

SOLUTIONS

Q1		/ 20
Q2		/ 15
Q3		/ 15
Q4		/ 15
Q5		/ 15
Q6		/ 20
Total		/ 100
Grade		

1. (20 points; 5 points for each part) Suppose an optimization problem in decision variables w_1 and w_2 has constraints

$$\begin{array}{rcrcl} -w_1 & + & w_2 & \leq & 7 \\ w_1 & & & \geq & 0. \end{array}$$

   (a) Devise a minimizing objective function for which the model has a unique optimal solution, and demonstrate that fact by solving the model graphically.

   (b) Devise a minimizing objective function for which the model has multiple optimal solutions, and demonstrate that fact by solving the model graphically.

   (c) Devise a minimizing objective function for which the model is unbounded, and demonstrate that fact by solving the model graphically.

   (d) Put your problem from part (b) into standard form.

For (a), (b), (c) see Version A.

(d) $\min \; w_1$

s.t. $-w_1 + w_2 \leq 7$

$w_1 \geq 0$

Equivalent to:

$\min \; w_1$

s.t. $-w_1 + w_2' - w_2'' + w_3 = 7$

$w_1, w_2', w_2'', w_3 \geq 0.$

2. (15 points) Formulate a linear programming problem to find a vector satisfying

$$3x_1 + x_2 \leq 5 \quad \text{and} \quad x_1 \geq 0,\ x_2 \geq 0$$

and having the maximum of

$$4x_1 - x_2 \quad \text{and} \quad -3x_1 + 2x_2$$

as small as possible.

$$\min_{x,z} \quad z$$

$$\text{s.t.} \quad z \geq 4x_1 - x_2$$

$$z \geq -3x_1 + 2x_2$$

$$3x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

3. (15 points) Jill van Rensselaer is taking courses in operations research, economics, statistics, and data structures. She has 40 study hours to prepare for her finals and wishes to divide her time to improve her grades as much as possible. Naturally, her favorite course is operations research, so she will spend as much time on it as on any other course. Still, she believes up to 15 hours of study could be useful in any of the courses, with each hour on operations research increasing her grade by 2%, each hour on economics yielding 4%, each on statistics producing 1%, and each on data structures adding 2%. Form a linear program to help Jill optimize her studying time.

x_{OR}, x_E, x_S, x_D = time spent on each course

$$\max \quad 2x_{OR} + 4x_E + x_S + 2x_D$$

s.t.

$$\begin{aligned} x_{OR} + x_E + x_S + x_D &\le 40 \\ x_{OR} - x_E &\ge 0 \\ x_{OR} - x_S &\ge 0 \\ x_{OR} - x_D &\ge 0 \end{aligned}$$

$$0 \le x_i \le 15, \quad \cancel{i = \ldots} \quad i = OR, \ldots, D$$

4. (15 points) Perform one simplex pivot in the following tableau:

$$
M = \begin{array}{c|ccccc|l}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & 2 & 0 & 7 & -2 & 0 & \\
\hline
20 & -4 & 0 & 3 & 2 & 1 & 20/2 \\
80 & 1 & 1 & 4 & 4 & 0 & 80/4 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
20 & -2 & 0 & 10 & 0 & 1 \\
\hline
10 & -2 & 0 & 3/2 & 1 & \frac{1}{2} \\
40 & 9 & 1 & -2 & 0 & -2 \\
\hline
\end{array}
$$

5. (15 points) Why can you conclude that the following problem has an unbounded optimal value? What is the corresponding simplex direction?

$$\begin{array}{lrcrcrcrcl}
\text{min} & & & 3x_2 & - & 2x_3 & & & & \\
\text{subject to} & & & 2x_2 & - & 3x_3 & + & x_4 & = & 6 \\
& x_1 & - & x_2 & - & 2x_3 & & & = & 5 \\
& & & & & & & x_i & \geq & 0, \quad i = 1, \ldots, 4
\end{array}$$

All entries in x_3 column of A are ≤ 0.

Simplex direction $\Delta x = (2, 0, 1, 3)$.

incoming variable

from column x_3 of A.

6. (20 points) The following linear program has multiple optimal solutions.

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{min} & & & x_2 & & & & & & & & & & \\
\text{subject to} & & - & 6x_2 & & & + & x_4 & - & 2x_5 & & & = & 6 \\
 & x_1 & + & x_2 & & & & & - & x_5 & & & = & 5 \\
 & & & 2x_2 & + & x_3 & & & + & 2x_5 & & & = & 6 \quad 6/2 \\
 & & - & 4x_2 & & & & & + & 4x_5 & + & x_6 & = & 16 \quad 16/4 \\
 & & & & & & & & & & & x_i & \geq & 0, \quad i = 1, \ldots, 6
\end{array}
$$

(a) (5 points) Give an optimal solution to the problem.

(b) (5 points) Find another optimal solution for the problem.

(c) (10 points) Graph the problem in an appropriate two dimensional space, and label the set of optimal solutions.

(a) $x_1 = 5,\ x_3 = 6,\ x_4 = 6,\ x_6 = 16,\ x_2 = x_5 = 0$

(b) Pivot x_5 into basis, pivot x_3 out of basis

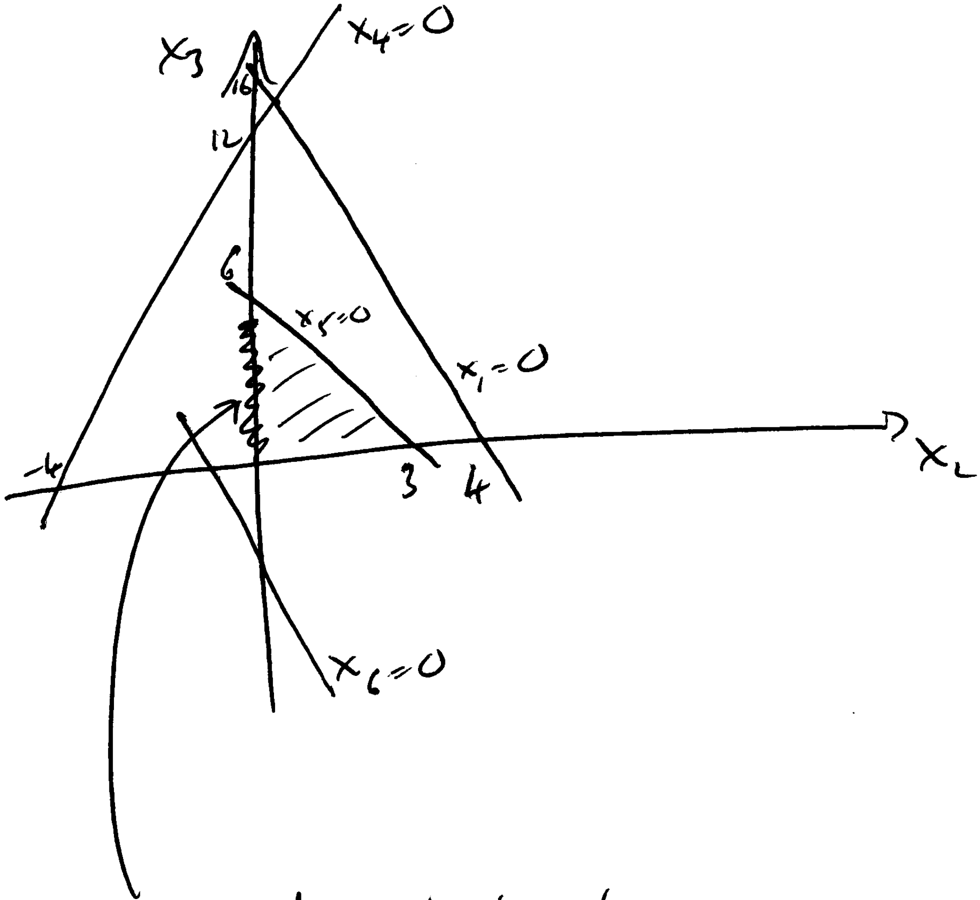
x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0	0	0	0	0
0	−4	1	1	0	0	12
1	2	$\frac{1}{2}$	0	0	0	8
0	1	$\frac{1}{2}$	0	1	0	3
0	−8	−2	0	0	1	4

Another optimal solution:

$x_1 = 8,\ x_4 = 12,\ x_5 = 3,\ x_6 = 4,\ x_2 = x_3 = 0$

(intentionally left blank)

(c) Graph in $x_2 x_3$-space (could also use $x_2 x_5$-space)



Set of optimal solutions.