A Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

First Exam, Friday, October 5, 2004.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts one hundred and ten minutes.

Q1	/ 30
Q2	/ 24
Q3	/ 25
Q4	/ 21
Total	/ 100
Grade	

SOLUTIONS

1. (30 points) The Tall Tree lumber company owns 95,000 acres of forestland in the Pacific northwest, at least 50,000 of which must be aerially sprayed for insects this year. Up to 40,000 acres could be handled by planes based at Squawking Eagle, and up to 30,000 acres could be handled from a more distant airstrip at Crooked Creek. Flying time, pilots and materials together cost \$3 per acre when spraying from Squawking Eagle and \$5 per acre when handled from Crooked Creek. Tall Tree seeks a minimum cost spraying plan.

   (a) (8 points) Formulate a mathematical programming model to select an optimal spraying plan using decision variables $x_1 :=$ thousands of acres sprayed from Squawking Eagle and $x_2 :=$ thousands of acres sprayed from Crooked Creek.

   (b) (8 points) Using a two-dimensional plot, solve your model graphically for an optimal spraying plan, and explain why it is unique.

   (c) (7 points) Put your model into standard form.

   (d) (7 points) On your two-dimensional plot, label each constraint as $v = 0$ for the appropriate variable v.

(a) min $3x_1 + 5x_2$

s.t. $50000 \le x_1 + x_2 \le 95000$

$0 \le x_1 \le 40000$

$0 \le x_2 \le 30000$

(c) min $3x_1 + 5x_2$

s.t. $x_1 + x_2 + x_3 = 95000$

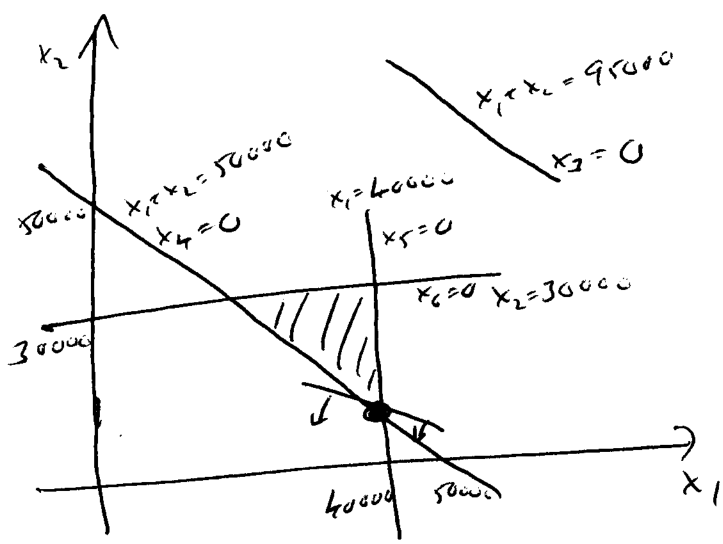
$x_1 + x_2 - x_4 = 50000$

$x_1 + x_5 = 40000$

$x_2 + x_6 = 30000$

$x_i \ge 0, \; i = 1, \dots, 6$

(b)



(d)

Optimal soln: $x_1 = 40000$, $x_2 = 10000$.

Cost: \$170000

Unique because contour uniquely touches feasible region here

2. (25 points. No partial credit. Each of parts (a)-(e) is worth 4 points, consisting of one point for each of a, b, c, and d.)

Consider the following tableau for a standard form linear program:

	x_1	x_2	x_3	x_4
3	0	a	0	1
b	1	c	0	3
4	0	d	1	1

(a) For what value(s) of a, b, c, d is this problem in optimal form?

(b) For what value(s) of a, b, c, d is this problem in unbounded form?

(c) For what value(s) of a, b, c, d is this problem infeasible?

(d) For what value(s) of a, b, c, d can we conclude from this tableau that this problem is degenerate?

(e) For what value(s) of a, b, c, d would x_2 replace x_1 in the basis?

(f) (5 points) For what value(s) of a, b, c, d can we conclude from this tableau that this problem has multiple optimal solutions? Consider the cases $b > 0$ and $b = 0$ separately.

(a) $a \geq 0$, $b \geq 0$, c = anything, d = anything

(b) $a < 0$, $b \geq 0$, $c \leq 0$, $d \leq 0$

(c) a = anything, $b < 0$, $c \geq 0$, d = anything

(d) a = anything, $b = 0$, c = anything, d = anything

(e) $a < 0$, $b \geq 0$, $c > 0$, either $d \leq 0$
or $d > 0$, $\frac{4}{d} \geq \frac{b}{c}$

(f) (i) $a = 0$, $b > 0$, c = anything, d = anything

(ii) $a = 0$, $b = 0$, $c \leq 0$, d = anything

3. (25 points)

   (a) Consider the following tableau for a standard form linear program:

$$M = \begin{array}{c|cccc|} & x_1 & x_2 & x_3 & x_4 \\ \hline 3 & 0 & -2 & 0 & 1 \\ \hline 2 & 1 & 1 & 0 & 3 \\ 4 & 0 & 4 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{c} \\ \\ 2/1 \\ 4/4 \end{array}$$

   i. (5 points) In the next simplex step, which nonbasic variable becomes basic and which basic variable becomes nonbasic?

   ii. (5 points) What is the simplex direction that you take from this tableau?

   iii. (5 points) Let M^1 denote the tableau arising after the simplex step. Find the pivot matrix P such that $M^1 = PM$.

(i) x_2 becomes basic

x_3 becomes nonbasic

(ii) $d = \begin{bmatrix} -1 \\ 1 \\ -4 \\ 0 \end{bmatrix}$

(iii) $P = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

(b) A standard form linear programming problem has tableau

$$M = \begin{array}{c|ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline 3 & 0 & -3 & 0 & 1 & -5 & 2 & 1 \\ \hline 4 & 1 & 1 & 0 & 3 & 2 & -1 & 1 \\ 9 & 0 & 2 & 1 & 1 & 3 & -1 & 4 \end{array}$$

The pivot matrix used for the next iteration is

$$P = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

i. (5 points) Calculate the resulting reduced costs and hence show that the new tableau is not optimal.

ii. (5 points) Find the pivot matrix for the subsequent iteration.

(i) $M' = PM = \begin{array}{c|ccccccc} & & & & & & \downarrow & \\ 15 & 3 & 0 & 0 & 10 & 1 & -1 & 4 \\ \hline 4 & & & & & & -1 & \\ 1 & & & & & & 1 & \end{array}$

(ii) $P' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

 4

4. (21 points)

(a) (8 points) Use the method of artificial variables to find a basic feasible solution to the linear program

$$\begin{array}{llcl}
\min & 3x_1 + x_2 - 4x_3 \cancel{+ x_4} & & \\
\text{subject to} & x_1 + x_2 + x_3 \cancel{- 2x_4} & = & 3 \\
& 2x_1 - x_2 + x_3 \cancel{+ 2x_4} & = & 6 \\
& x_i & \geq & 0 \quad \text{for } i = 1, \ldots, 4.
\end{array}$$

0	0	0	0	~~0~~	1	1
3	1	1	1	~~-2~~	1	0
6	2	-1	1	~~2~~	0	1

Get canonical form

	-9	-3	0	-2	~~0~~	0	0
3/1	3	1	1	1	~~1~~	1	0
6/2	6	2	-1	1	~~2~~	0	1

0	0	3	1	~~?~~	3	0
3	1	1	1	~~?~~	1	0
0	0	-3	-1	~~?~~	-2	1

Optimal to artificial problem.

η still basic.

Delete artificial columns,

and pivot on 2nd constraint:

0	0	0	0
3	1	-2	0
0	0	3	1

BFS:

$x_1 = 3$, $x_3 = 0$ (basic)

$x_2 = 0$ (nonbasic)

(b) (12 points) The optimal tableau for the artificial problem constructed when solving the linear program

$$
\begin{array}{lrcrcrcrcrcl}
\text{min} & 8x_1 & + & x_2 & + & 2x_3 & + & 3\cancel{4}x_4 & + & 4x_5 & & \\
\text{subject to} & 5x_1 & + & x_2 & + & 2x_3 & + & 4x_4 & + & 2x_5 & = & 9 \\
 & 4x_1 & + & x_2 & + & x_3 & + & 3x_3 & + & 3x_3 & = & 7 \\
 & & & & & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 4.
\end{array} \qquad (P)
$$

is

$$
M = \begin{array}{c|cccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_1 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
 & 2 & 1 & 0 & 1 & 1 & -1 & 1 & -1 \\
 & 5 & 3 & 1 & 0 & 2 & 4 & -1 & 2
\end{array}
$$

where y_1 and y_2 are the artificial variables. Find an optimal solution to (P).

Initial tableau:

	x_1	x_2	x_3	x_4	x_5
0	8	1	2	$3\cancel{4}$	4
2	1	0	1	1	-1
5	3	1	0	2	4

Get canonical form:

	-9	3	0	0	-1	2
2/1	2	1	0	1	1	-1
5/2	5	3	1	0	2	4

-7	4	0	1	0	1
2	1	0	1	1	-1
1	1	1	-2	0	2

Optimal soln:

$x_2 = 1, \; x_4 = 2$

$x_1 = x_3 = x_5 = 0$

Value = 7.

Check: Feasible ✓

Value ✓