IV Fourier Series

and Applications to P.D.E.

A. Fourier Series

Def: In the space $V$ of integrable functions on $[-L,L]$, we define the inner product by

$$<f,g> = \int_{-L}^{L} f(x)g(x)\,dx$$

We define the norm of $f$ by

$$\|f\| = \left[ \int_{-L}^{L} f(x)^2\,dx \right]^{1/2}$$

Def: orthogonal set, orthonormal set

a) $(g_n)$ is said to be orthogonal if

$$\int_{-L}^{L} g_n(x)g_m(x)\,dx = 0, \quad n \neq m$$

b) Suppose $(\phi_n)$ is orthogonal and such that

$$\|\phi_n\| = 1, \quad \forall n,$$

then $(\phi_n)$ is said to be orthonormal.

Example 1:

Let $g_0(x) = 1$

$g_1(x) = \cos \frac{\pi}{L} x$, $g_2(x) = \sin \frac{\pi}{L} x$

$g_{2n-1}(x) = \cos \frac{n\pi}{L} x$, $g_{2n}(x) = \sin \frac{n\pi}{L} x$

then $(g_n)$ is orthogonal
Example 2

Let \( \phi_0(x) = \frac{1}{\sqrt{2L}} \)

\[ \phi_n(x) = \frac{g_n(x)}{\sqrt{L}}, \quad n > 1 \]

then \( (\phi_n) \) is orthonormal.

**Def:** Fourier series

Suppose \( f \) integrable on \([\!-\!L, L]\)

Let

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \]

then the series

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right]
\]

is called Fourier series of \( f \),

and \( a_n, b_n \) the Fourier coefficients,

and we write

\[ f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right] \]

**Def:** We say that \( (S_n) \) converges in the mean to \( f \)

if \( \| S_n - f \| \to 0 \), \( n \to \infty \)

**Notation:** \( S_n \to f \) l.i.m.

**Note:** If \( S_n(x) \) denote the partial sum of the Fourier series of \( f \), we always have convergence in the mean:

\( S_n \to f \) l.i.m.

whenever \( f \) is integrable. But this does not imply pointwise convergence.
Example 3

\[ f(x) = \begin{cases} 
1 & , 0 < x < \pi \\
0 & , -\pi < x < 0 
\end{cases} \]

\[ a_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos(nx) \, dx = \begin{cases} 
1 & n=1 \\
0 & , n > 1 
\end{cases} \]

\[ b_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin(nx) \, dx = \frac{1}{\pi} \left( \frac{\cos n\pi}{n} \right) \bigg|_{0}^{\pi} = \frac{1}{n\pi} (1 - (-1)^n) \]

\[ = \begin{cases} 
0 & , n \text{ even} \\
\frac{2}{n\pi} & , n \text{ odd} 
\end{cases} \]

\[ f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{L} \]

Thm:

Suppose \( f \) and \( f' \) piecewise continuous on \( [-L, L] \).
Then

a) At a point of continuity \( x_0 \), we have

\[ f(x_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi}{L} x_0 + b_n \sin \frac{n\pi}{L} x_0 \right] \]

b) At a point of discontinuity \( x_0 \), we have

\[ \frac{f(x_0^+) + f(x_0^-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi}{L} x_0 + b_n \sin \frac{n\pi}{L} x_0 \right] \]

Note: In the previous example \( x = 0 \), is a point of discontinuity. Therefore, the sum of the Fourier series gives the value \( \frac{1}{2} \) which is the average (or the middle value):

\[ \frac{1 + 0}{2} \]

However, for points \( 0 < |x| < \pi \), the sum coincides with the value of \( f \) (case a) of the previous thm.)
Even and odd functions

a) If \( f \) is even, then
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x
\]
where \( a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi}{L} x \, dx \)

b) If \( f \) is odd, then
\[
f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x
\]
where \( b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \)

Example 4
\[
f(x) = x, \quad -L < x < L
\]
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x, \quad -L < x < L
\]
\[
b_n = \frac{2}{L} \int_{0}^{L} x \sin \frac{n\pi}{L} x \, dx
\]
\[
= \frac{2}{n\pi} \left[ \frac{x \cos \frac{n\pi}{L} x}{L} \right]_{0}^{L} + \frac{1}{n\pi} \int_{0}^{L} \cos \frac{n\pi}{L} x \, dx
\]
\[
= \frac{2}{n\pi} \left( -1 \right)^{n-1}
\]
x = \frac{2}{n\pi} \sum_{n=1}^{\infty} \left( -1 \right)^{n-1} \frac{\sin \frac{n\pi}{L} x}{n}, \quad -L < x < L

Example 5
\[
f(x) = 1 - x^2, \quad -1 < x < 1
\]
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x, \quad -1 < x < 1
\]
a_0 = 2 \int_{0}^{1} (1-x^2) \, dx = 2 \left( 1 - \frac{x^3}{3} \right) \bigg|_{0}^{1} = \frac{4}{3}
\]
a_n = 2 \int_{0}^{1} (1-x^2) \cos \frac{n\pi}{L} x \, dx
\]
\[
= \frac{4}{(n\pi)^2} \left( -1 \right)^{n-1}
\]
Therefore

\[ 1-x^2 = -\frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos n\pi x \quad , \quad -1 \leq x \leq 1 \]

**Def:** Suppose \( f \) is integrable on \([0, L] \)

a) We define the Fourier cosine series of \( f \) by

\[ f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \]

where \( a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \)

as the even extension of \( f \) on \([-L, L] \)

b) We define the Fourier sine series of \( f \) by

\[ f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \]

where \( b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \)

as the odd extension of \( f \) on \([-L, L] \)

**Note:**

If \( f \) is integrable on \([0, T] \), and periodic of period \( T > 0 \), then the corresponding formula for its Fourier series is:

\[ f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{2\pi n x}{T} + b_n \sin \frac{2\pi n x}{T} \right] \]

\[ a_n = \frac{2}{T} \int_{0}^{T} f(x) \cos \frac{2\pi n x}{T} \, dx \]

\[ b_n = \frac{2}{T} \int_{0}^{T} f(x) \sin \frac{2\pi n x}{T} \, dx \]
Complex form of Fourier Series

Suppose \( f \) is integrable on \([-L, L]\).

The series

\[
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n x}{L}}
\]

where
\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{2\pi n x}{L}} \, dx
\]

is called the Fourier series \( \text{in complex form of } f \).

Example 6

Expand \( e^x \) \( \text{in complex form of the Fourier series for } -\pi < x < \pi \).

\[
e^x = \sum_{n=-\infty}^{\infty} c_n e^{i \pi n x}, \quad -\pi < x < \pi
\]

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} e^{-i \pi n x} \, dx
\]

\[
c_n = \frac{1}{2\pi} \left. \frac{e^{-x} e^{-i \pi n x}}{-\pi (1 + in)} \right|_{-\pi}^{\pi}
\]

\[
c_n = \frac{\sinh \pi}{\pi} \left( \frac{(-1)^n}{1 + in} \right)
\]

\[
e^x = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1 + n^2} e^{i \pi n x}
\]
Consider the heat equation

\[ \alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 \]  

(1)

**B.c.** \[ u(0,t) = 0 = u(L,t) \]  

(2)

**I.c.** \[ u(x,0) = f(x) \]  

(3)

We use the method of separation of variables:

\[ u(x,t) = X(x) T(t) \]

We get from the P.D.E.:

\[ \alpha^2 X'' T = X T' \]

\[ \frac{X''}{X} = \frac{T'}{\kappa^2 T} = \text{constant} = -\lambda \]

\[ X'' + \lambda X = 0 \]

\[ X(0) = 0 = X(L) \]

(4)

\[ T' + \lambda \kappa^2 T = 0 \]

(5)

The B.V.P. (4) yields

\[ X_n = \sin n \frac{\pi}{L} x \]

\[ n = 1, 2, \ldots \]

for \( \lambda_n = \left( \frac{n \pi}{L} \right)^2 \)

(5) \[ \Rightarrow \quad T_n = e^{-\left( \frac{n \pi}{L} \kappa \right)^2 t} \]

By the superposition principle

\[ u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left( \frac{n \pi}{L} \kappa \right)^2 t} \sin n \frac{\pi}{L} x \]

(6)
To find \( c_n \), put \( t = 0 \)

\[
f(x) = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin n \frac{\pi}{L} x
\]

i.e. \( c_n \) are the Fourier sine series coefficients of \( f \)

\[
c_n = \frac{2}{L} \int_0^L f(x) \sin n \frac{\pi}{L} x \, dx
\]

Up to this point, we have only a formal process. To obtain a complete justification of the solution more analysis is needed and conditions on \( f \) have to be imposed as seen in the problem of convergence of Fourier series.

**Application** : \( L = 50 \text{ cm} \)

\[
u(x, 0) = f(x) = 20^\circ \text{ C}
\]

\[
u(x, t) = 80 \sum_{m=1}^{\infty} \frac{\sin (2m-1) \frac{\pi x}{50}}{2m-1} \exp (-[2(m-1) \frac{\pi t}{50}]^2 t)
\]

**Case of insulated boundaries**

Consider the heat equation

\[
\kappa^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 \tag{1}
\]

B.C \quad \[
u_x (0, t) = 0 = \nu_x (L, t) \tag{2'}
\]

I.C \quad \[
u(x, 0) = f(x) \tag{3}
\]

Following the same procedure, we arrive at the B.V.P.

\[
X'' + \lambda X = 0 \quad X'(0) = 0 = X'(L)
\]

whose solutions are \( X_n = \cos n \frac{\pi x}{L} \), \( n = 0, 1, 2, \ldots \)

and we obtain the series
\[ u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{L} \kappa\right)^2 t} \cos \frac{n\pi}{L} x \]

The coefficients \( a_n \) can be found by putting \( t = 0 \)

\[ f(x) = u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x , \]

\[ \Rightarrow a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \]

i.e. \( a_n \) are the Fourier cosine series coefficients of \( f \)

**Application**: \( L = 50 \text{ cm} \)

\[ u(x,t) = f(x) = 20 \degree C \]

\[ a_0 = 40 \quad a_{n/2} = 20 \]

\[ a_n = \frac{40}{L} \int_0^L \cos \frac{n\pi}{L} x \, dx = 0 \quad n > 1 \]

\[ \therefore u(x,t) = 20 \quad 0 < x < L, \ t > 0 \]

**Vibrations of a string**

Consider the wave equation of a vibrating string attached at both ends.

\[ \kappa^2 u_{xx} = u_{tt} , \quad 0 < x < L , \ t > 0 \]

**B.C.** \[ u(0,t) = u(L,t) = 0, \ t > 0 \]

**I.C.** \[ \begin{cases} u(x,0) = f(x) & \text{initial displacement} \\ u_t(x,0) = g(x) & \text{velocity} \end{cases} \]
We use the method of separation of variables

\[ u(x, t) = X(x) T(t) \]

From the ODE, we get

\[ \kappa^2 X'' T = X T' \]

\[ \frac{X''}{X} = \frac{T'}{\kappa^2 T} = \text{constant} \]

\[ \Rightarrow \quad X'' + \lambda X = 0, \quad X(0) = 0 = X(L) \]

\[ T'' + \lambda \kappa^2 T = 0 \]

\[ \Rightarrow \quad X_n = \sin n \frac{\pi}{L} x, \quad \lambda_n = \left(\frac{n \pi}{L}\right)^2 \]

\[ T_n = A_n \cos n \frac{\pi}{L} x + B_n \sin n \frac{\pi}{L} x \]

\[ \Rightarrow \quad u_n = \left( A_n \cos n \frac{\pi}{L} x + B_n \sin n \frac{\pi}{L} x \right) \sin n \frac{\pi}{L} x \]

By the superposition principle

\[ u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos n \frac{\pi}{L} x + B_n \sin n \frac{\pi}{L} x \right) \sin n \frac{\pi}{L} x \]

We can get the coefficients \( A_n, B_n \) from the I.C. at \( t=0 \)

\[ f(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin n \frac{\pi}{L} x \]

\[ \Rightarrow \quad A_n = \frac{2}{L} \int_0^L f(x) \sin n \frac{\pi}{L} x \, dx \]

\[ g(x) = u_t(x, 0) = \sum_{n=1}^{\infty} \left( \frac{n \pi}{L} \right) B_n \sin n \frac{\pi}{L} x \]

\[ \Rightarrow \quad \frac{n \pi}{L} B_n = \frac{2}{L} \int_0^L g(x) \sin n \frac{\pi}{L} x \, dx \]

\[ \Rightarrow \quad B_n = \frac{2}{n \pi \kappa} \int_0^L g(x) \sin n \frac{\pi}{L} x \, dx \]
Applications of Fourier Series

To Laplace equation

case of a rectangle

\[ u_{xx} + u_{yy} = 0 \quad 0 < x < a \quad 0 < y < b \]
\[ u(x, 0) = 0 = u(x, b) \]
\[ u(0, y) = 0, \quad u(a, y) = f(y) \]
\[ f(0) = 0 = f(b) \]

The separation of variables method could have been used as done before. But let us use the formal Fourier series expansion of \( u \)

\[ u(x, y) = \sum_{n=1}^{\infty} C_n(x) \sin \frac{n\pi}{b} y \]

From the D.E., we get

\[ C_n'' - \left( \frac{n\pi}{b} \right)^2 C_n = 0 \quad n = 1, 2, \ldots \]

\[ \Rightarrow \]

\[ C_n = B_n \cosh \frac{n\pi}{b} x + A_n \sinh \frac{n\pi}{b} x \]

\[ C_n(0) = 0 \Rightarrow B_n = 0 \]

\[ \Rightarrow \]

\[ u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{b} \cosh \frac{n\pi}{b} x \sin \frac{n\pi}{b} y \]

We can obtain \( A_n \) from the last B.C. at \( x = a \)

\[ f(y) = u(a, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{b} a \sin \frac{n\pi}{b} y \]

\[ \Rightarrow \]

\[ A_n \sinh \frac{n\pi}{b} a = \frac{2}{b} \int_{0}^{b} f(y) \sin \frac{n\pi}{b} y \, dy = b_n \]

Therefore

\[ u(x, y) = \sum_{n=1}^{\infty} b_n \frac{\sinh \frac{n\pi}{b} x}{\sinh \frac{n\pi}{b} a} \sin \frac{n\pi}{b} y \]
Application:

\[ a = 3, \quad b = 2 \]

\[ f(\gamma) = \begin{cases} 
    \gamma, & 0 \leq \gamma \leq 1 \\
    2 - \gamma, & 1 \leq \gamma \leq 2 
\end{cases} \]

\[ u(x, y) = \frac{8}{\pi^2} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\sin n\pi x/2}{n^2} \frac{\sin n\pi y}{\sinh(3n\pi/2)} \]

Case of a disc

Consider Laplace equation in polar coordinates

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad r < a \]

B.C. \[ u(a, \theta) = f(\theta), \quad 0 \leq \theta < 2\pi \]

\[ u \text{ bounded} \]

By a formal Fourier series expansion of \( u \), we write

\[ u(r, \theta) = \frac{A_0(r)}{2} + \sum_{n=1}^{\infty} \left[ A_n(r) \cos n\theta + B_n(r) \sin n\theta \right] \]

From the D.E.

\[ A_0'' + \frac{1}{r} A_0' = 0, \quad (\Rightarrow A_0 = d_1 + d_2 \ln r) \]

and both \( A_n, B_n \) will satisfy the Euler type D.E.

\[ y'' + \frac{1}{r} y' - \frac{n^2}{r^2} y = 0 \]

\[ y = c_1 r^n + c_2 r^{-n} \]

Since our solution must be bounded, \( c_2 = 0 \) and

\[ A_n = \text{constant} \]

Therefore we can write:

\[ u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \left[ a_n \cos n\theta + b_n \sin n\theta \right] \]
To obtain the coefficients $a_n$, $b_n$, we use the B.C at $r = a$.

$$f(\theta) = u(a, \theta) = \frac{a}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos n\theta + b_n \sin n\theta \right],$$

hence

$$a_n = \frac{i}{\pi} \int_{0}^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$b_n = \frac{i}{\pi} \int_{0}^{2\pi} f(\theta) \sin n\theta \, d\theta$$

**Note:** Exterior B.V.P.

For the B.V.P.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad r > a$$

$$u(\theta) = f(\theta)$$

$u$ bounded

we get

$$u(r, \theta) = \frac{a}{2} + \sum_{n=1}^{\infty} \left( \frac{a}{r} \right)^n \left[ a_n \cos n\theta + b_n \sin n\theta \right]$$

where $a_n$, $b_n$ are given by the same formulae.