II.1 General properties of linear second order linear D.E.

In order to understand the basic methods for solving second order linear D.E., we introduce some fundamental properties which will be used repeatedly in this chapter.

We consider the D.E.

\[ y'' + a(x)y' + b(x)y = g(x) \]

where \( a, b \) and \( g \) are continuous on \( I \).

Denote by \( L \) the differential operator

\[ L(y) = y'' + a(x)y' + b(x)y \]

\[ L(y) = 0 \] is the homogeneous D.E.

\[ L(y) = g \] " nonhomogeneous "

Note: \( L \) is linear

This means \( L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) \),

for any scalar \( c_1 \) and \( c_2 \).

Property 1: Superposition principle

If \( y_1 \) and \( y_2 \) are 2 solutions of \( L(y) = 0 \),

then \( c_1 y_1 + c_2 y_2 \) is also a solution.

\[ L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) = 0 \]
Property 2: Existence and uniqueness of solution

Based on a similar theorem for existence and uniqueness in chapter 1, the I.V.P.

\[ \begin{cases} L(y) = 0 \\ y(x_0) = y_0, \ y'(x_0) = y'_0 \end{cases}, \quad x_0 \in I \]

has a unique solution.

Def: Linear independence and linear dependence

\( \varphi_1 \) and \( \varphi_2 \) are said to be linearly independent on \( I \)

if

\[ c_1 \varphi_1(x) + c_2 \varphi_2(x) = 0 \quad \forall x \text{ in } I \]

implies \( c_1 = 0 = c_2 \)

If \( \varphi_1 \) and \( \varphi_2 \) are not linearly independent, we say that they are linearly dependent.

Notation: L.I. for linearly independent
L.D. "" dependent

Example: \( \varphi_1(x) = \cos x \), \( \varphi_2(x) = \sin x \) are L.I.

\[ c_1 \cos x + c_2 \sin x = 0 \quad \forall x \]

\[ \Rightarrow c_1 = 0 \quad (x = 0) \]

and \( c_2 = 0 \quad (x = \frac{\pi}{2}) \)

Property 3: There exist 2 L.I. solutions of \( L(y) = 0 \)

[This is a consequence of property 2]

Remark: This simply says that the space of solutions has a basis.
Def: A set of 2 linearly independent solutions of \( L(y) = 0 \) is called a **basis** or a fundamental set of solutions.

**Property 4:** If \( \{ y_1, y_2 \} \) is a basis of \( L(y) = 0 \), then every solution has the form
\[
y = c_1 y_1 + c_2 y_2
\]

Def: **Wronskian**

The Wronskian of \( \phi_1 \) and \( \phi_2 \) is defined by the determinant
\[
W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi'_1(x) & \phi'_2(x) \end{vmatrix} = \phi_1(x)\phi'_2(x) - \phi'_1(x)\phi_2(x).
\]

**Notations:** \( W(\phi_1, \phi_2)(x) \), \( W(\phi_1, \phi_2) \), \( W(x) \), \( W' \)

**Abel's Formula**

If \( y_1 \) and \( y_2 \) are 2 solutions of \( L(y) = 0 \), then
\[
W(y_1, y_2)(x) = W(y_1, y_2)(x_0) \exp \left(-\int_{x_0}^x a(t) \, dt \right)
\]

If we first show that \( W' + a(x)W = 0 \),

\[
W' = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -a_1y_1' - b_1y_1 & -a_1y_2' - b_1y_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -a_1y_1' & -a_1y_2' \end{vmatrix} = -a_1W
\]

By solving the D.E. for \( W \), we obtain the formula

**Remark:** \( W(x_0) \neq 0 \Rightarrow W(x) \neq 0 \ \forall x \)
Consequence of Abel's formula:

Suppose that $\gamma_1$ and $\gamma_2$ are 2 solutions of $L(\gamma) = 0$.

\[ W(\gamma_1, \gamma_2)(x_0) \neq 0 \iff \gamma_1 \text{ and } \gamma_2 \text{ are L.I.} \]

By the previous remark $W(x_0) \neq 0 \Rightarrow W(x) \neq 0$. Suppose \( c_1 \gamma_1 + c_2 \gamma_2 = 0 \) which implies \( c_1 \gamma_1' + c_2 \gamma_2' = 0 \).

It is known from linear algebra that the system:

\[
\begin{cases}
    c_1 \gamma_1 + c_2 \gamma_2 = 0 \\
    c_1 \gamma_1' + c_2 \gamma_2' = 0
\end{cases}
\]

has the trivial solution \( (c_1, c_2) = 0 \) \( \iff \) \( W(x) \neq 0 \) i.e. \( W(x_0) \neq 0 \iff \gamma_1, \gamma_2 \text{ L.I.} \).

A special method for obtaining a second L.I. solution, or the reduction of order formula:

Consider the D.E. \( L(\gamma) = \gamma''' + a(x)\gamma' + b(x)\gamma = 0 \) where \( a \) and \( b \) are continuous on \( I \).

If \( \gamma_1(x) \neq 0 \) is a solution, then \( \gamma_2(x) = \gamma_1(x) \int \frac{e^{-\int a(x)dx}}{\gamma_1^2(x)} dx \)

is a second solution L.I. of \( \gamma \).

\[
\begin{align*}
\gamma_2 &= \gamma_1 u \\
\gamma_2' &= \gamma_1' u + \gamma_1 u' \\
\gamma_2'' &= \gamma_1'' u + 2\gamma_1' u' + \gamma_1 u''
\end{align*}
\]

\[ 0 = L(\gamma_2) = \gamma_2'' + a(\gamma_2') + b\gamma_2 = u L(\gamma_1) + \gamma_1 u'' + (a(\gamma_1') + a_1) u' \]

\[ \Rightarrow u'' + (a + \frac{2\gamma_1'}{\gamma_1}) u' = 0 \Rightarrow u' = \gamma_1^{-2}(x) e^{-\int a(x)dx} \]

\[ \Rightarrow \gamma_2 = \gamma_1(x) \int \gamma_1^{-2}(x) \exp(-\int a(x)dx) dx \]

\[ W(\gamma_1, \gamma_2)(x) = \exp(-\int a(x)dx) \neq 0 \Rightarrow \text{L.I.} \]
Example

\[ x^2 y'' + 3x y' + 2y = 0, \quad x > 0. \]

Suppose \( y_1(x) = x^{-1} \cos(\ln x) \) is a solution.

Then \( y_2(x) = y_1(x) \int \frac{e^{-3\frac{\ln x}{x}}}{\gamma_1^2(x)} \, dx \)

\[ = y_1(x) \int \frac{x^{-3} \cos^2(\ln x)}{x^{-2} \cos^2(\ln x)} \, dx \]

\[ = y_1(x) \int \frac{dx/x}{\cos^2(\ln x)} \]

\[ = y_1(x) \tan(\ln x) \]

\[ = x^{-1} \sin(\ln x) \]
So far, we have considered properties of homogeneous d.e. Now we arrive at the last property which is related to nonhomogeneous d.e.

**Property 5**

Consider \( L(y) = g \) (N.H.)

If
a) \( y_p \) is a particular solution of (N.H.)
b) \( \{y_1, y_2\} \) is a basis for \( L(y) = 0 \)

then every solution of (N.H.) has the form:
\[
y = c_1 y_1 + c_2 y_2 + y_p
\]

Let \( y \) be any solution of \( L(y) = g \).

By the linearity of \( L \), we have:
\[
L(y - y_p) = L(y) - L(y_p) = g - g = 0
\]
\[
\therefore y - y_p = c_1 y_1 + c_2 y_2
\]
\[
\Rightarrow y = c_1 y_1 + c_2 y_2 + y_p \]

**Example:** \( y'' + y = \cos 2x \)

\( y_1 = \cos x \), \( y_2 = \sin x \)
\( y_p = -\frac{1}{3} \cos 2x \)
\( y = c_1 \cos x + c_2 \sin x - \frac{1}{3} \cos 2x \)
II. 2 Second order linear D.E. of constant coefficients

\[ A : L(y) = 0 \]

Consider \( L(y) = y'' + ay' + by = 0 \), where \( a, b \) are constants.

Preliminary remark: \( L(e^{rx}) = (r^2 + ar + b)e^{rx} \), where \( r \) is a scalar, real or complex

Def: \( p(r) = r^2 + ar + b \) is called the characteristic polynomial of \( L \), and \( r^2 + ar + b = 0 \), the characteristic equation.

Thm: Consider \( L(y) = y'' + ay' + by = 0 \)
where \( a \) and \( b \) are constants,
and let \( p(r) = r^2 + ar + b \)

\[ \begin{align*}
\text{Case 1:} & \quad r_1, r_2 \text{ are distinct roots of } p \\
& \text{then } y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \text{ in the general solution}
\end{align*} \]

\[ \begin{align*}
\text{Case 2:} & \quad r_1 \text{ is a repeated root of } p \\
& \text{then } y = c_1 x e^{r_1 x} + c_2 e^{r_1 x} \text{ in the general solution}
\end{align*} \]

\[ \begin{align*}
\text{Case 3:} & \quad r_1 = \alpha + j\beta \text{ is complex, then } \text{then } \\
y & = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \text{ in the general solution}
\end{align*} \]

We have \( L(e^{rx}) = p(r)e^{rx} \) which implies

by differentiating with respect to \( r \), \( L(xe^{rx}) = (p(r) + xp(r))e^{rx} \)

In case 1, putting \( r = r_1 \) and \( r = r_2 \) in the first equation, we get
\( L(e^{r_1 x}) = 0 \) \( , \) \( L(e^{r_2 x}) = 0 \)
\( \Rightarrow y_1 = e^{r_1 x}, y_2 = e^{r_2 x} \) are solutions. To show L.I., we compute the wronskian at \( x = 0 \), \( W(y_1, y_2)(0) = r_2 - r_1 \neq 0 \).
Therefore \( \{ \gamma_1, \gamma_2 \} \) is a basis. Thus \( \gamma = c_1 \gamma_1 + c_2 \gamma_2 \)

**Case 2.** Putting \( r = r_1 \) in the second equation, we have \( L(xe^{r_1 x}) = 0 \). \( \gamma_1 = e^{r_1 x}, \gamma_2 = xe^{r_1 x} \) are 2 solutions. For L.I., we look at the Wronskian at \( x = 0 \)

\[ W(\gamma_1, \gamma_2)(0) = 1 \neq 0 \Rightarrow \{ e^{r_1 x}, xe^{r_1 x} \} \text{ is a basis}. \]

Therefore \( \gamma = c_1 \gamma_1 + c_2 \gamma_2 \)

**Case 3:** when \( r_1 = \alpha + i\beta \)

We already know that \( L(e^{r_1 x}) = 0 \)

\[ 0 = L(e^{(\alpha + i\beta)x}) = L \left[ e^{\alpha x} e^{i\beta x} \right] \]

\[ = L(e^{\alpha x} \cos \beta x) + i L(e^{\alpha x} \sin \beta x) \]

\[ = \gamma_1 = e^{\alpha x} \cos \beta x, \gamma_2 = e^{\alpha x} \sin \beta x \text{ are solutions} \]

\[ W(\gamma_1, \gamma_2)(0) = \beta \neq 0 \Rightarrow \text{L.I.} \]

Therefore \( \gamma = c_1 \gamma_1 + c_2 \gamma_2 \)

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**Example 1:** \( \gamma'' + 5\gamma' + 6\gamma = 0 \)

\[ \rho(r) = r^2 + 5r + 6 = 0 \Rightarrow r_1 = -2, r_2 = -3 \]

\( \gamma = c_1 e^{-2x} + c_2 e^{-3x} \)

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**Example 2:** \( \gamma'' + 3\gamma' + 2\gamma = 0 \)

\[ \rho(r) = r^2 + 3r + 2 = 0 \Rightarrow r_1 = -1, r_2 = -2 \]

\( \gamma = c_1 e^{-x} + c_2 e^{-2x} \)

\[ \begin{cases} c_1 + c_2 = 1 \\ -c_1 - 2c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_2 = -1 \\ c_1 = 2 \end{cases} \]

\( \gamma = 2e^{-x} - e^{-2x} \)
Example 3: \( y'' - 2y' + y = 0 \), \( y(0) = 1 \), \( y'(0) = -1 \)
\[ r^2 - 2r + 1 = 0 \quad r = 1 = 1 \]
\( y = c_1 e^x + c_2 x e^x \)
\( 1 = y(0) = c_1 \)
\(-1 = y'(0) = 1 + c_2 \quad \Rightarrow c_2 = -2 \)
\( \therefore y = (1 - 2x) e^x \)

Example 4: \( y'' + 4y' + 5y = 0 \), \( y(0) = 1, y'(0) = -1 \)
\[ r^2 + 4r + 5 = 0 \Rightarrow \text{roots: } -2 \pm i \]
\( y = e^{-2x} (c_1 \cos x + c_2 \sin x) \)
\( 1 = y(0) = c_1 \)
\(-1 = y'(0) = -2c_1 + c_2 = -2 + c_2 \quad \Rightarrow c_2 = 1 \)
\( \therefore y = e^{-2x} (\cos x + \sin x) = \sqrt{2} e^{-2x} \cos (x - \frac{\pi}{4}) \)

This is an example of damped vibration

Example 5: \( y'' + \omega^2 y = 0 \), \( \omega > 0 \), \( y(0) = y_o \), \( y'(0) = y'_o \)
\[ r^2 + \omega^2 = 0 \Rightarrow \text{roots: } \pm i \omega \]
\( y = c_1 \cos \omega x + c_2 \sin \omega x \)
\( y_o = y(0) = c_1 \)
\( y'_o = y'(0) = \omega c_2 \quad \Rightarrow c_2 = \frac{y'_o}{\omega} \)
\( \therefore y = y_o \cos \omega x + \frac{y'_o}{\omega} \sin \omega x \)

This is an example of a simple harmonic oscillator
In a similar way one can consider $n$th order linear D.E. of constant coefficients.

Consider the homogeneous D.E.

$$L(y) = y^{(n)} + q_1 y^{(n-1)} + \ldots + q_n y = 0$$

where $q_1, \ldots, q_n$ are constants.

Let $p(r) = r^n + q_1 r^{n-1} + \ldots + q_n$ be the characteristic polynomial.

If $r_1, \ldots, r_s$ are the distinct roots of $p$ and $m_1, \ldots, m_s$ their respective multiplicities,
then the following set of functions:

$$\left\{ e^{r_1 x}, x e^{r_1 x}, \ldots, x^{m_1-1} e^{r_1 x} \right\}$$

$$\left\{ e^{r_2 x}, x e^{r_2 x}, \ldots, x^{m_2-1} e^{r_2 x} \right\}$$

$$\left\{ e^{r_3 x}, x e^{r_3 x}, \ldots, x^{m_3-1} e^{r_3 x} \right\}$$

is a basis.

Example

$$D^2 (D^2 - \omega^2)(D^2 + \omega^2) y = 0$$

where $D = \frac{d}{dx}$, $\omega > 0$

$$p(r) = r^2 (r^2 - \omega^2)(r^2 + \omega^2)$$

$r_1 = 0$  \hspace{1cm} $m_1 = 2$$

$r_3 = -\omega$  \hspace{1cm} $r_4 = \omega$  \hspace{1cm} $m_3 = 1$  \hspace{1cm} $m_4 = 1$$

$r_5 = i\omega$  \hspace{1cm} $r_6 = -i\omega$  \hspace{1cm} $m_5 = 1$  \hspace{1cm} $m_6 = 1$$

The general solution is:

$$y = a_1 + a_2 x + b_1 e^{\omega x} + b_2 e^{-\omega x} + c_1 \cos \omega x + c_2 \sin \omega x$$
The method of undetermined coefficients

Let us start with a simple problem:

Example: Find a particular solution of the (N.H.) D.E.

\[ L(y) = y'' + y = e^{-2x} \quad \text{(N.H.)} \]

If we denote by \( D \), the derivative operator with respect to \( x \), then we remark that

\[ (D + 2I) e^{-2x} = 0 \]

Therefore \( \text{(N.H.) becomes} \)

\[ (D + 2I)(D^2 + I) y = 0, \]

which is a homogeneous linear D.E. of constant coefficients. We have seen how to solve this type of D.E.

\[ y = q_1 \cos x + q_2 \sin x + C e^{-2x}, \quad q_1, q_2, C = \text{constants} \]

since \( q_1 \cos x + q_2 \sin x = y_c(x) \) is the general solution of the homogeneous equation \( y'' + y = 0 \), the particular solution for \( \text{(N.H.)} \) must have the form \( C e^{-2x} \). By substitution into \( \text{(N.H.)} \), we can determine the constant \( C \).

\[ e^{-2x} = L(C e^{-2x}) = 5C e^{-2x} \]

\[ 5C = 1 \quad \Rightarrow \quad C = \frac{1}{5} \quad \Rightarrow \quad y_p = \frac{1}{5} e^{-2x} \]

Therefore \( y = q_1 \cos x + q_2 \sin x + 5^{-1} e^{-2x} \)

is the general solution of \( \text{(N.H.)} \).
Now consider the general problem
\[ L(y) = g \quad (N.H.) \]
where \( L \) is a differential operator of constant coefficients. If the differential operator of constant coefficients \( N \) is such that
\[ N(g) = 0, \]
then we say that \( N \) annihilates \( g \). In this case, \( (N.H.) \) becomes
\[ N(L(y)) = N(g) = 0 \]
which is a homogeneous linear D.E. of constant coefficients, which we can solve. Therefore, we will be able to know exactly the form of the particular solution of \( (N.H.) \). The coefficients will then be found by substitution into \( (N.H.) \). This is called the method of undetermined coefficients based on the annihilator approach.

**Basic annihilators**
\[ D = \frac{d}{dx} \]

1. \( D^m \) annihilates: \( 1, x, \ldots, x^{m-1} \)

2. \( (D-\lambda)^m \) annihilates: \( e^{\lambda x}, xe^{\lambda x}, \ldots, x^{m-1}e^{\lambda x} \)

3. \( [(D-\alpha)^2 + \beta^2]^m \) annihilates: \( e^{\alpha x}\{\cos \beta x, \ldots, x^{m-1}e^{\alpha x}\sin \beta x} \)
I. By Leibniz rule

\[ D^m (e^{-\lambda x} y) = e^{-\lambda x} \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^k (y^{m-k}) \]

\[ = e^{-\lambda x} (D-\lambda)^m y \]

which implies the shift rule:

\[ (D-\lambda)^m y = e^{\lambda x} D^m (e^{-\lambda x} y) \]

By application of this rule, we get:

\[ (D-\lambda)^m \left[ e^{\lambda x} \left( c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} \right) \right] \]

\[ = e^{\lambda x} D^m \left( c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} \right) = 0 \]

3. In the complex form of 2" (\( \lambda = \alpha + i\beta \))

Example 1

Find an annihilator for \( \sin^2 x \)

Since \( \sin^2 x = \frac{1}{2} - \cos 2x \)

and \( D(1) = 0 \), \( (D^2 + 4) \cos 2x = 0 \)

\( D(D^2 + 4) \) annihilates \( \sin^2 x \)

Example 2

Find an annihilator for \( x^2 e^{-x} + x \cos x = g(x) \)

Since \( (D+1)^3 x^2 e^{-x} = 0 \)

and \( (D^2 + 1)^2 x \cos x = 0 \)

\( (D+1)^3 (D^2 + 1)^2 \) annihilates \( g \)
Applications

Example 3

\[ y'' - y = x^2 \]
\[ D^3(x^2) = 0 \]
\[ D^3(b^2 - 1)y = 0 \]
\[ y = \frac{c_0 + c_1 x + c_2 x^2 + 2e^{-x} + 2e^x}{y_p} \]
\[ y_c \]

\[ y_p(x) = c_0 + c_1 x + c_2 x^2 \]
\[ x^2 = L(y_p) = 2c_2 - (c_0 + c_1 x + c_2 x^2) \]
\[ = (2c_2 - c_0) - c_1 x - c_2 x^2 \]
\[ \Rightarrow c_2 = -1 \quad c_1 = 0 \quad 2c_2 - c_0 = 0 \Rightarrow c_0 = -2 \]
\[ \therefore y_p(x) = -2 - x^2 \]
\[ \text{and} \quad y = q_1 e^{-x} + q_2 e^x - 2 - x^2 \]

Example 4

\[ y'' + y = xe^x \]

Since \((b-1)^2 x e^x = 0\), and \(p(r) = r^2 + 1\)

the form of the particular solution is

\[ y_p(x) = (c_0 + c_1 x) e^x \]

\[ xe^x = L(y_p) = \left[ 2(c_1 e^x + (c_0 + c_1 x)e^x) \right] \]
\[ = \left[ (2c_1 + c_0) + 2c_1 x \right] e^x \]
\[ 2c_1 = 1 \quad 2(c_1 + c_0) = 0 \Rightarrow c_0 = -\frac{1}{2}, c_1 = \frac{1}{2} \]

\[ y_p(x) = \frac{1}{2} (x-1) e^x \]
\[ y = q_1 e^{3x} + q_2 \sin x + \frac{1}{2} (x-1) e^x \]
Thm:

Consider the D.E.
\[ L(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = g(x)e^{\lambda x} \quad (N.H.) \]

where \( a_1, \ldots, a_n \) are constants

and \( g(x) = b_0 + b_1 x + \ldots + b_k x^k \)

Let \( p(r) = r^n + a_1 r^{n-1} + \ldots + a_n \)

Case 1: \( \lambda \) is not a root of \( p \)

There is a particular solution having the form:
\[ \psi_p(x) = (c_0 + c_1 x + \ldots + c_k x^k)e^{\lambda x} \]

Case 2: \( \lambda \) is equal to one of the roots of \( p \) (with multiplicity \( m \))

There is a particular solution having the form:
\[ \psi_p(x) = (c_0 + c_1 x + \ldots + c_k x^k)x^m e^{\lambda x} \]

In both cases the coefficients can be obtained by substitution into the \((N.H.)\) equation

Example 5
\[ y'' - 2y' + y = (1+x)e^x \]
\[ p(r) = r^2 - 2r + 1 = (r - 1)^2 \Rightarrow \lambda = 1, m = 2 \]

This is case 2:
\[ \psi_p(x) = (c_0 + c_1 x)x^2 e^x = (c_0 x^2 + c_1 x^3)e^x \]
\[ (1+x)e^x = L(\psi_p) = (2c_0 + c_1 x)e^x \Rightarrow c_0 = \frac{1}{2}, c_1 = \frac{1}{2} \]
\[ \therefore \psi_p(x) = \left( \frac{1}{2} + \frac{x}{2} \right)x^2 e^x \]

Example 6
\[ y'' + 2y' + y = \cos x \]
\[ p(r) = (r + 1)^2 \]

This is case 1:
\[ \psi_p(x) = A \cos x + B \sin x \]
\[ \cos x = L(\psi_p) = -2A \sin x + 2B \cos x \Rightarrow A = 0, B = \frac{1}{2} \]
\[ \therefore \psi_p(x) = \frac{1}{2} \sin x \]
The method of variation of parameters

Thm:

Consider the D.E.

\[ L(\gamma) = \gamma'' + a(x)\gamma' + b(x)\gamma = g(x) \]  \hspace{1cm} (N.H.)

where \(a, b\) and \(g\) are continuous on \(I\).

If \(\phi_1\) and \(\phi_2\) are 2 L.I. solutions of \(L(\gamma) = 0\), then

\[ \gamma_p(x) = \int_{x_0}^{x} \frac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{W(\phi_1, \phi_2)(t)} g(t) \, dt \quad \text{for } x_0 \in I \]

is a particular solution of \(\text{(N.H.)}\).

II. We look for a solution of \(\text{(N.H.)}\) having the form:

\[ \gamma_p(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) \]

According to Lagrange, if the following system

\[
\begin{align*}
    u_1'\phi_1 + u_2'\phi_2 &= 0 \\
    u_1'\phi_1' + u_2'\phi_2' &= g
\end{align*}
\]

is satisfied, then 

\[ L(\gamma_p) = g : \]

\[
L(u_1\phi_1 + u_2\phi_2) = (u_1\phi_1 + u_2\phi_2)'' + a(u_1\phi_1 + u_2\phi_2)' + b(u_1\phi_1 + u_2\phi_2) = u_1\phi_1'' + u_2\phi_2'' + u_1\phi_1' + u_2\phi_2' + u_1a\phi_1' + u_2a\phi_2' + u_1b\phi_1 + u_2b\phi_2 = u_1L(\phi_1) + u_2L(\phi_2) + g
\]

Solving the system \(\text{(S)}\): \(u_1' = -\frac{\phi_2}{W}g\), \(u_2' = \frac{\phi_1}{W}g\)

\[ u_1\phi_1 + u_2\phi_2 = \phi_1(x)\int -\frac{\phi_2(x)}{W(x)} g(x) \, dx + \phi_2(x)\int \frac{\phi_1(x)}{W(x)} g(x) \, dx \]
Example 1

\[ y'' + \omega^2 y = g(t), \quad \omega > 0 \]

\[ y(0) = y_0 \]

\[ y'(0) = \dot{y}_0 \]

\[ \phi_1(x) = \cos \omega x \quad , \quad \phi_2(x) = \sin \omega x \]

\[ W(x) = \left| \begin{array}{cc} \phi_1' & \phi_2' \\ \phi_1 & \phi_2 \end{array} \right| = \omega \]

According to the formula

\[ y_p(x) = \int_0^x \frac{\cos \omega t \sin \omega x - \cos \omega x \sin \omega t}{\omega} g(t) \, dt \]

\[ = \frac{1}{\omega} \int_0^x \sin \omega (x-t) g(t) \, dt \]

The general solution is:

\[ c_1 \phi_1(x) + c_2 \phi_2(x) + y_p(x) \]

\[ y(0) = y_0 \quad \Rightarrow \quad c_1 = y_0 \]

\[ y'(0) = \dot{y}_0 \quad \Rightarrow \quad c_2 = \frac{\dot{y}_0}{\omega} \]

\[ y = y_0 \cos \omega x + \frac{\dot{y}_0}{\omega} \sin \omega x + \frac{1}{\omega} \int_0^x \sin \omega (x-t) g(t) \, dt \]

Example 2

\[ y'' - \frac{2}{x^2} y = x^{-1}, \quad x > 0 \]

\[ \phi_1(x) = x^2, \quad \phi_2(x) = x^{-1} \quad \Rightarrow \quad W = -3 \]

\[ u_1' = -\frac{x^{-1}}{-3} = \frac{x^2}{3} \quad \Rightarrow \quad u_1 = -\frac{x^2}{3} \]

\[ u_2' = \frac{x^2}{-3} x^{-1} = -\frac{x}{3} \quad \Rightarrow \quad u_2 = -\frac{x^2}{6} \]

\[ y_p = u_1 \phi_1 + u_2 \phi_2 = -\frac{x^2}{3} - \frac{x}{6} = -\frac{x}{2} \]

The general solution is:

\[ c_1 x^2 + c_2 x^{-1} - \frac{x}{2} \]
B Euler equation

\[ \text{Def: } L(y) = x^2 y'' + ax y' + by = 0, \quad x \neq 0, \]

where \( a \) and \( b \) are constants, is called Euler equation.

Preliminary remark

\[ L(x^r) = [r(r-1) + ar + b]x^r, \quad x > 0 \]

\[ \begin{align*}
  (x^r)'' &= r(r-1)x^{r-2} \\
  (x^r)' &= r x^{r-1} \\
  \Rightarrow L(x^r) &= (r(r-1) + ar + b)x^r 
\end{align*} \]

\[ \text{Def: } g(r) = r(r-1) + ar + b \quad \text{is called the indicial polynomial of } L \]

We have \[ L(x^r) = g(r)x^r \quad x > 0 \]

Consequence:

\[ L(x^r/\ln x) = [g'(r) + g(r)/\ln x]x^r, \quad x > 0 \]

\[ \text{This is obtained by differentiation with respect to } r \]
Thm

Consider Euler equation

\[ L(y) = x^2 y'' + ax y' + by = 0, \quad x > 0 \]

where \( a \) and \( b \) are constants, and let \( g(r) = r(r-1) + ar + b \) be the indicial polynomial.

**Case 1:** \( r_1, r_2 \) distinct roots of \( g \)

then \( y = c_1 x^{r_1} + c_2 x^{r_2} \) is the general solution

**Case 2:** \( r_1 \) is a repeated root of \( g \)

then \( y = c_1 x^{r_1} + c_2 x^{r_1} \ln x \) is the general solution

**Case 3:** \( r_1 = k + i\beta \) is a complex root of \( g \)

then \( y = x^k \left( c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x) \right) \)

is the general solution

\[ \begin{align*}
\text{case 1:} & \quad \text{Put } r = r_1 \text{ and } r = r_2 \text{ in } L(x^r) = g(r)x^r, \\
& \text{this implies that } x^{r_1}, x^{r_2} \text{ are solutions. Also} \\
& W(x^{r_1}, x^{r_2}) |_{x=1} = r_2 - r_1 \neq 0 \Rightarrow \text{L.I.}
\end{align*} \]

\[ \text{case 2:} \quad \text{Put } r = r_1 \text{ in } L(x^{r} \ln x) = (g(r) + g(r) \ln x)x^r, \]

we get \( x^{r_1} \ln x \) another solution. For L.I.

\[ W(1) = 1 \neq 0 \Rightarrow \text{L.I.} \]

\[ \text{case 3:} \quad \text{By the real and imaginary part of} \]

\( x^{k+\beta} \), \( x^k \cos(\beta \ln x), x^k \sin(\beta \ln x) \)

They are also L.I. by considering the Wronskian at \( x=1 \) \( W(1) = \beta \neq 0 \)
Example 1

\[ x^2 y'' + x y' - n^2 y = 0 \quad , \quad x > 0 \quad , \quad n > 0 \]

\[ g(r) = r(r-1) + r - n^2 = r^2 - n^2 \]

\[ r_1 = n \quad , \quad r_2 = -n \]

\[ y = c_1 x^n + c_2 x^{-n} \quad , \quad x > 0 \]

Example 2

\[ x^2 y'' + 3x y' + y = 0 \quad , \quad x > 0 \]

\[ g(r) = r(r-1) + 3r + 1 = r^2 + 2r + 1 = (r+1)^2 \]

\[ r_1 = -1 \quad \text{a repeated root} \]

\[ y = c_1 x^{-1} + c_2 x^{-1} \ln x \quad , \quad x > 0 \]

Example 3

\[ x^2 y'' + 2x y' + y = 0 \quad , \quad x > 0 \]

\[ g(r) = r(r-1) + 2r + 1 = r^2 + r + 1 \]

\[ r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \]

\[ y = c_1 x^{-\frac{1}{2}} \cos \left( \frac{\sqrt{3}}{2} \ln x \right) + c_2 x^{-\frac{1}{2}} \sin \left( \frac{\sqrt{3}}{2} \ln x \right) \]