13. \( y'' - 2y' - 3y = -3te^{-t} \). Solve for the general solution.

The homogeneous problem \( y'' - 2y' - 3y = 0 \) has characteristic polynomial \( r^2 - 2r - 3 = (r-3)(r+1) \). This implies homogeneous solution \( y_h = Ae^{3t} + Be^{-t} \).

Considering the characteristic polynomial and the Right Hand Side of the equation we can write the particular solution \( y_p = Cte^{-t} + Dte^{-t} \).

Solving the equation \(-3te^{-t} = y'' + 2y' - 3y\), we get the values of \( C \) and \( D \).

\[
-3te^{-t} = y'' + 2y' - 3y_p = (-2Ce^{-t} + Cte^{-t} + 2De^{-t} - 4Dte^{-t} + Dte^{-t})
\]

\[
+ (-2Ce^{-t} + 2Cte^{-t} - 4Dte^{-t} + 2Dte^{-t}) + (-3Ce^{-t} - 3Dte^{-t})
\]

\[
= -4Ce^{-t} + 2Dte^{-t} - 8Dte^{-t}
\]

\[
\Rightarrow -3te^{-t} = -8Dte^{-t} \quad \text{and} \quad 0 = -4Ce^{-t} + 2Dte^{-t}
\]

\[
\Rightarrow D = \frac{3}{8} \quad \text{and} \quad C = \frac{3}{16}
\]

Now we can write the general solution as

\[
y_g = y_h + y_p = Ae^{3t} + Be^{-t} + \frac{3}{16}te^{-t} + \frac{3}{8} t^2 e^{-t} \square
\]
\[6y'' + 2y' + y = 2e^{-t}. \text{ Solve for the general solution.}\]

The homogeneous differential equation \(y'' + 2y' + y = 0\) has characteristic polynomial \(r^2 + 2r + 1 = (r+1)^2\). Which implies homogeneous solution \(y_h = Ae^{-t} + Be^{-t}\).

Our particular solution has form \(y_p = Ct^2e^{-t}\).

\[2e^{-t} = y_p'' + 2y_p' + y_p = (Ct^2e^{-t}) + (-2Ct^2e^{-t} + 4Cte^{-t}) + (Ct^2e^{-t} - 4Cte^{-t} + 2Ce^{-t}) = 2Ce^{-t}\]

\[\Rightarrow C = 1. \Rightarrow y_p = t^2e^{-t}\]

Therefore we have general solution

\[y_g = y_h + y_p = Ae^{-t} + Be^{-t} + t^2e^{-t}\]

\[q\]

\[u'' + \omega_0^2 u = \cos \omega t. \text{ Solve for the general solution.}\]

The homogeneous problem \(u'' + \omega_0^2 u = 0\) has characteristic polynomial \((r^2 + \omega_0^2) = (r + i\omega_0)(r - i\omega_0)\). Which implies homogeneous solution \(y_h = A\cos \omega_0 t + B\sin \omega_0 t\). Our particular solution has form \(y_p = C\cos \omega t\).

\[\cos \omega t = y_p'' + \omega_0^2 y_p = -C\omega^2 \cos \omega t + \omega_0^2 C \cos \omega t = C(\omega_0^2 - \omega)\cos \omega t\]

\[\Rightarrow C = \frac{1}{\omega_0^2 - \omega}. \text{ Thus we have general solution}\]

\[y_g = y_h + y_p = A\cos \omega_0 t + B\sin \omega_0 t + \frac{1}{\omega_0^2 - \omega^2} \cos \omega t\]

\[\square\]
13 Solve the Initial Value Problem: \( y'' + y' - 2y = 2t \), \( y(0) = 0 \), \( y'(0) = 1 \).

The homogeneous problem has characteristic polynomial

\[ r^2 + r - 2 = (r + 2)(r - 1) \]

which implies \( y_h = Ae^{-2t} + Be^t \).

The particular solution has the form \( y_p = Ct + D \).

\[ 2t = y_p'' + y_p' - 2y = C - 2(Ct + D) \]

\[ \Rightarrow C - 2D = 0 \text{ and } -2Ct = 2t \Rightarrow C = -1 \text{ and } D = -\frac{1}{2} \]

Thus \( y_g = Ae^{-2t} + Be^t - t - \frac{1}{2} \).

Now we can use the Initial Conditions to find the values of \( A \) and \( B \).

\[ 0 = y_g(0) = A + B - \frac{1}{2} \text{ and } 1 = y_g'(0) = -2A + B - 1 \]

\[ \Rightarrow A = -\frac{1}{2} \text{ and } B = 1 \]

Therefore, the solution to the initial value problem is

\[ y(t) = \frac{-1}{2} e^{-2t} + e^t - t - \frac{1}{2} \]