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Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

Third Exam, Friday, December 6, 2002.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of six questions, and lasts one hundred minutes.

You can collect the graded exam from me next week. Grades should be available early next week.

SOLUTIONS.

Q1	/ 15
Q2	/ 20
Q3	/ 10
Q4	/ 20
Q5	/ 15
Q6	/ 20
Total	/ 100
Grade	

1. (15 points) Determine the best bound on the optimal solution value of an ILP with each of the following objective functions that is available from the specified LP relaxation optima \tilde{x} .

(a) $\min x_1 - 4x_2 + 70x_3, \tilde{x} = (1, 0, \frac{2}{7})$.

(b) $\min 60x_1 - 16x_2 + 10x_3, \tilde{x} = (\frac{1}{2}, 1, \frac{1}{2})$.

(c) $\max 90x_1 + 11x_2 + 30x_3, \tilde{x} = (0, \frac{1}{2}, 5)$.

(a) Optimal value ≥ 21

(b) Optimal value ≥ 19

(c) Optimal value ≤ 155.5

2. (20 points) Jane is applying to graduate school. There are five schools she is interested in attending, but she can only afford to send application fees to three of them. She has put a value on the education at each of the schools, and she has also decided what her chances of being accepted are. She wants to maximize the sum of the values of the three chosen schools, but she wants to choose the three in such a way that the sum of the probabilities of rejection is no greater than 1.5. The values are as follows:

School n	1	2	3	4	5
Value v_n	90	150	80	100	120
Probability of rejection p_n	0.4	0.7	0.4	0.5	0.6

- (a) (4 point) Formulate Jane's problem as an integer programming problem, with one constraint for the number of applications and one constraint for the probabilities.
- (b) (8 points) How would you solve this integer program using dynamic programming and backward recursion equations? Describe the states, stages and recursion equations. State $f_5(s)$ explicitly. (Hint: You will probably need two state variables.)
- (c) (8 points) Illustrate your solution method by solving the following subproblem: There are two schools 4 and 5 left, Jane has one application left, and she only has 0.5 of slack remaining in her probability constraint.

$$\begin{aligned}
 \text{(a) } \max & \sum_{n=1}^5 v_n x_n \\
 \text{s.t.} & \sum_{n=1}^5 p_n x_n \leq 1.5 \\
 & \sum_{n=1}^5 x_n \leq 3 \\
 & x_n \text{ binary, } n=1, \dots, 5
 \end{aligned}$$

- (b) At stage n : choose value of x_n .
 2 dimensional state variable: $s =$ remaining slack in prob constraint
 $t =$ remaining slack in rec'd constraint

$$f_n(s, t, x_n) = v_n x_n + f_{n+1}(s - p_n x_n, t - x_n)$$

$$f_n(s, t) = \max_{x_n} f_n(s, t, x_n)$$

$$f_5(s, t) = \begin{cases} v_5 & \text{if } s \geq p_5 \text{ and } t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 120 & \text{if } s \geq 0.6 \text{ and } t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{(c) } f_4(0.5, 1):$$

$$f_4(0.5, 1, 0) = f_5(0.5, 1) = 0$$

$$f_4(0.5, 1, 1) = 100 + f_5(0, 0) = 100$$

$$\text{So: } x_4^* = 1, x_5^* = 0,$$

$$\text{value} = 100$$

3. (10 points) The point $x = (1, 3, 2)$ is an interior point for the linear program

$$\begin{aligned} \min \quad & 10x_1 + 7x_2 + 6x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 = 13 \\ & x_1 + x_2 + x_3 = 6 \\ & x_i \geq 0 \quad i = 1, 2, 3 \end{aligned}$$

Let $\mu = 6$. Show there exists a dual feasible solution y with dual slacks s satisfying $x_i s_i = \mu$ for $i = 1, 2, 3$. (Hint: The dual of $\min\{c^T x : Ax = b, x \geq 0\}$ is $\max\{b^T y : A^T y \leq c\}$.)

Dual:

$$\begin{aligned} \max \quad & 13y_1 + 6y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 + s_1 = 10 \\ & 3y_1 + y_2 + s_2 = 7 \\ & y_1 + y_2 + s_3 = 6 \\ & s_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$

Need:

$$\begin{aligned} s_1 = \mu, \quad \text{so} \quad \underline{s_1 = 6} \\ 3s_2 = \mu, \quad \text{so} \quad \underline{s_2 = 2} \\ 2s_3 = \mu, \quad \text{so} \quad \underline{s_3 = 3} \end{aligned}$$

Need:

$$\begin{aligned} 2y_1 + y_2 + 6 = 10 & \quad \left. \begin{array}{l} 2y_1 + y_2 = 4 \quad \textcircled{1} \\ 3y_1 + y_2 = 5 \quad \textcircled{2} \\ y_1 + y_2 = 3 \quad \textcircled{3} \end{array} \right\} \end{aligned}$$

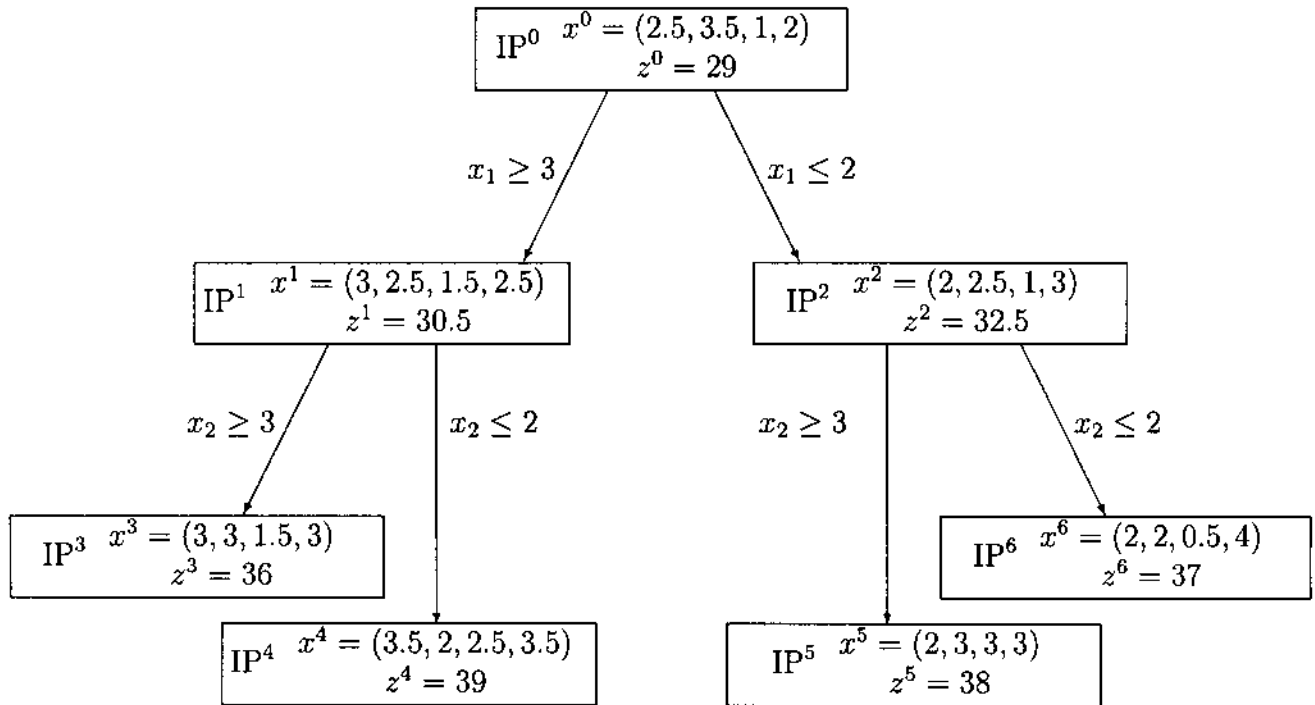
$\textcircled{1} - \textcircled{3} \Rightarrow \boxed{y_1 = 1}$ Substitute in $\textcircled{1}$: $\boxed{y_2 = 2}$

Check $\textcircled{1}, \textcircled{2}, \textcircled{3}$: All ok, so we have $x_i s_i = \mu$ for $i = 1, 2, 3$.

4. (20 points) The following integer programming problem is being solved using branch and bound:

$$\begin{aligned} \min \quad & x_1 + 3x_2 + 2x_3 + 7x_4 \\ \text{subject to} \quad & Ax = b \\ & x_i \geq 0, i = 1, \dots, 4 \\ & x_i \text{ integer}, i = 1, \dots, 4, \end{aligned}$$

where $Ax = b$ is a given set of linear constraints. The progress so far is shown. Here, x^i is the solution to the LP relaxation at node IP^i and z^i is the value of x^i .



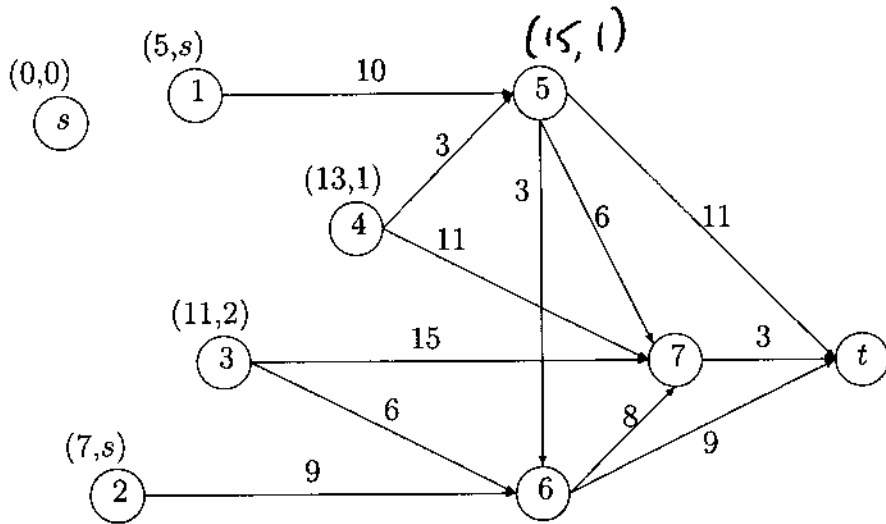
- (a) (5 points) What is the best solution found so far to the integer program?
- (b) (8 points) Which node(s) of the branch and bound tree can be fathomed?
- (c) (7 points) What subproblems would you create next?

(a) Only integer solution found so far is x^5 , so this is best

(b) Know optimal value is ≤ 38 . Fathom IP^4 by bounds
Fathom IP^5 by integrality.

(c) Split IP^3 since that has the best value.
Divide into $x_3 \leq 1$ and $x_3 \geq 2$.

5. (15 points) In the graph below, we want to find the shortest path from vertex s to vertex t . Using Dijkstra's algorithm, so far we have labeled vertices s , 1, 2, 3, and 4. For each labeled vertex, the first label gives the length of the shortest path to that vertex from s going via labeled vertices only, and the second label gives the index of the previous vertex on this shortest path. The edge length is indicated next to each edge. Note that the edges between labeled vertices have not been drawn in the picture. What is the shortest path from s to t and what is its length?



U	1	2	3	4	5	6	7	8	t
$s, 1, 2, 3, 4$	5, s	7, s	11, 2	13, 1	15, 1	16, 2	24, 4		∞
$s, 1, 2, 3, 4, 5$	—	—	—	—	↑	16, 2	21, 5		26, 5
$s, 1, 2, 3, 4, 5, 6$	—	—	—	—	—	↑	21, 5		25, 6
$s, 1, 2, 3, 4, 5, 6, 7$	—	—	—	—	—	—	↑		24, 7

Shortest path has length 24.

Path is (tracing backwards): $s, 1, 5, 7, t$

6. (20 points) In this problem, we are considering a standard form primal-dual pair,

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \quad (P) \qquad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + s = c \\ & s \geq 0 \end{array} \quad (D)$$

(a) (10 points) The diagonal scaling matrix that we use for primal-dual interior methods is D , where the i th diagonal entry is $D_{ii} = \sqrt{\frac{x_i}{s_i}}$. Some interior point methods use a diagonal matrix X with $X_{ii} = x_i$. Other methods use a diagonal matrix S with $S_{ii} = \frac{1}{s_i}$. Show that $D_{ii} = \sqrt{\frac{1}{\mu}} X_{ii} = \sqrt{\mu} S_{ii}$ if $x_i s_i = \mu$ for all i for some $\mu > 0$.

(b) (10 points) Show that if $\bar{x} > 0$ and $\left(\frac{x_i - \bar{x}_i}{\bar{x}_i}\right)^2 \leq 1$ then $x_i \geq 0$.

(a) $D_{ii} = \sqrt{\frac{x_i}{s_i}}$ is given.

Also given: $x_i s_i = \mu$ for all i .

Then: (i) $\sqrt{\frac{1}{\mu}} X_{ii} = \sqrt{\frac{1}{x_i s_i}} X_{ii} = \sqrt{\frac{x_i}{s_i}} = D_{ii} \checkmark$

(ii) $\sqrt{\mu} S_{ii} = \sqrt{x_i s_i} \cdot \frac{1}{s_i} = \sqrt{\frac{x_i}{s_i}} = D_{ii} \checkmark$

(b) Given: $\bar{x} > 0$, $\left(\frac{x_i - \bar{x}_i}{\bar{x}_i}\right)^2 \leq 1$.

~~Sol.~~ $\left(\frac{x_i - \bar{x}_i}{\bar{x}_i}\right)^2 \leq 1$

Now, $\left(\frac{x_i - \bar{x}_i}{\bar{x}_i}\right)^2 \leq 1 \iff \left|\frac{x_i}{\bar{x}_i} - 1\right| \leq 1$

$\iff \frac{x_i}{\bar{x}_i} - 1 = \pm 1 \iff x_i = 0 \text{ or } 2\bar{x}_i$

Since it is a positive quadratic, we have $\left(\frac{x_i - \bar{x}_i}{\bar{x}_i}\right)^2 \leq 1 \implies 0 \leq x_i \leq 2\bar{x}_i$.

