A

Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

Second Exam, Friday, September 19, 2003.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of six questions, and lasts one hundred minutes.

SOLUTIONS.

Q1	/ 20
Q2	/ 15
Q3	/ 15
Q4	/ 15
Q5	/ 15
Q6	/ 20
Total	/ 100
Grade	

1. (20 points; 10 points for each part) Each of the following is a linear program with no optimal solution. State the corresponding dual problem, solve both primal and dual graphically, and verify that whenever primal or dual is unbounded, the other is infeasible.

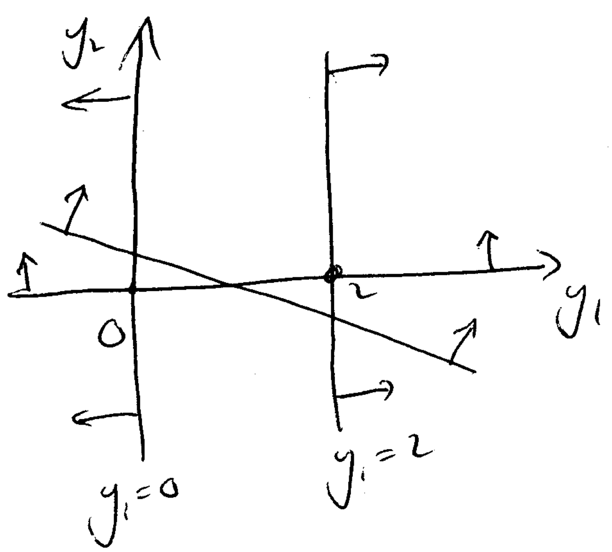
   (a)

$$\begin{array}{llcl} \max & 4x_1 + x_2 & & \\ \text{subject to} & 2x_1 + x_2 & \geq & 4 \\ & 3x_2 & \leq & 12 \\ & x_1, x_2 & \geq & 0 \end{array} \quad (P)$$

Dual:

$$\begin{array}{llcl} \min & 4y_1 + 12y_2 & & \\ \text{s.t.} & 2y_1 & \geqslant & 4 \\ & y_1 + 3y_2 & \geqslant & 1 \\ & y_1 \leq 0, \; y_2 \geqslant 0 & & \end{array} \quad (D)$$

Plot (D):

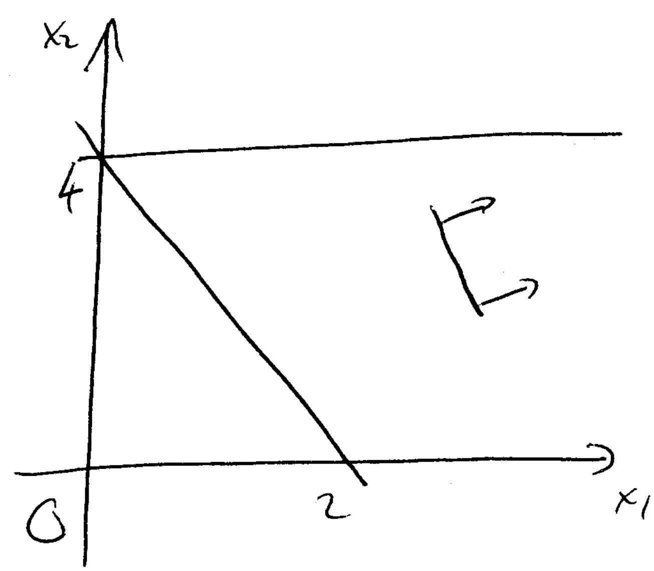


$y_1 \leq 0, \; 2y_1 \geqslant 4$

contradict each other

(D) infeasible

Plot (P)



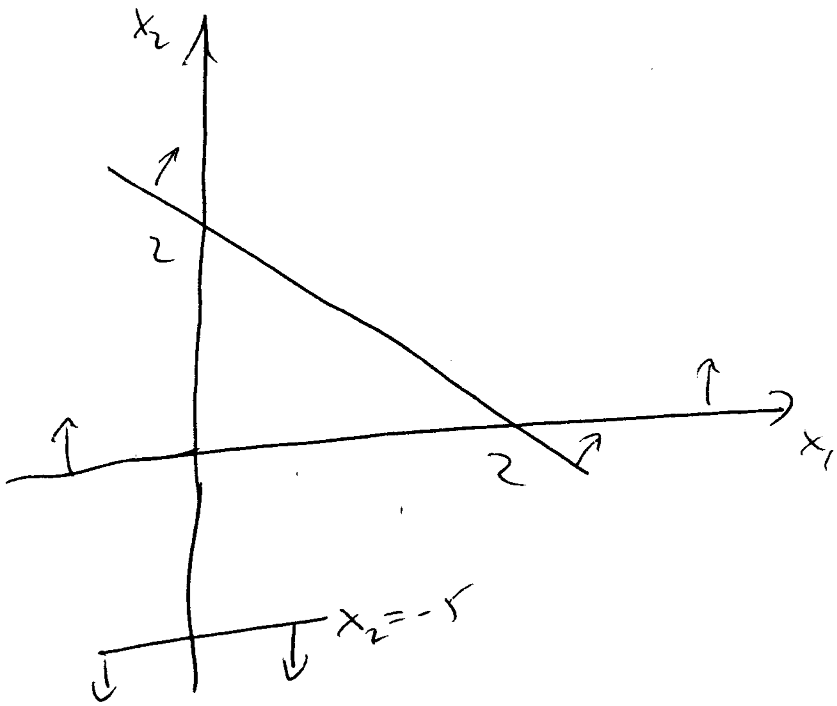
Primal is unbounded.

Ray: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b)

$$
\begin{array}{llrcrcl}
\text{min} & & 10x_1 & + & 3x_2 & & \\
\text{subject to} & & x_1 & + & x_2 & \geq & 2 \\
& & & & -x_2 & \geq & 5 \\
& & & & x_1, x_2 & \geq & 0
\end{array}
\quad (P)
$$

Plot (P).

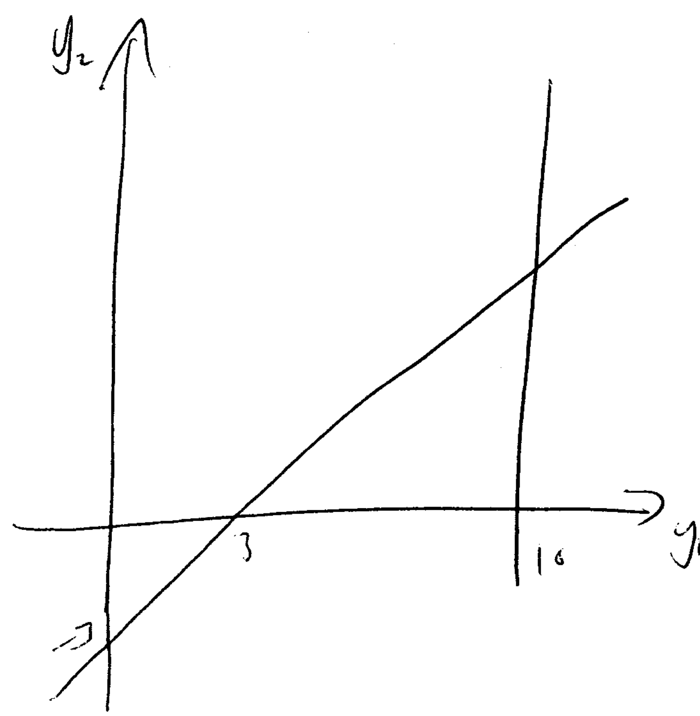


Infeasible.
Can't have $x_2 \geq 0$
and $-x_2 \geq 5$.

Dual:

$$
\begin{array}{lrcl}
\max & 2y_1 + 5y_2 & & \\
\text{s.t.} & y_1 & \leq & 10 \\
& y_1 - y_2 & \leq & 3 \\
& y_1, y_2 & \geq & 0
\end{array}
$$

Unbounded.
Ray: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



2. (15 points) The optimal solution to the linear programming problem

$$\begin{array}{lrcrcrcl} \max & -4x_1 & + & 3x_2 & + & x_3 & & \\ \text{subject to} & -x_1 & + & x_2 & + & 2x_3 & \leq & 5 \quad (1) \\ & x_1 & + & x_2 & + & x_3 & \leq & 9 \quad (2) \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

is $x_1 = 0,\ x_2 = 5,\ x_3 = 0$.

(a) (5 points) Give a dual problem to this linear program.

(b) (10 points) Use complementary slackness to find an optimal solution to your dual problem.

Dual:

$$\begin{array}{lrcrl} \min & 5y_1 & + & 9y_2 & \\ \text{s.t.} & -y_1 & + & y_2 & \geq -4 \\ & y_1 & + & y_2 & \geq 3 \\ & 2y_1 & + & y_2 & \geq 1 \\ & & & y_1, y_2 & \geq 0 \end{array}$$

(b) Given primal solution shows we need $y_1 + y_2 = 3$ (3).

Also, given primal solution gives a slack of 4 in constraint (2).

So we need $y_2 = 0$ (4)

(3), (4) $\Rightarrow$ $\boxed{y_1 = 3,\ y_2 = 0}$

Check: This is feasible in dual ✓.

Value: 15.

Primal value: 15

Good.

3. (15 points) Consider the linear programming minimization problem represented by the following tableau M_0:

$$M_0 = \begin{array}{|c|ccccccc|} \hline 0 & 0 & 1 & 0 & 0 & 5 & 1 & -1 \\ \hline 7 & 1 & -1 & 0 & 0 & -3 & 7 & 2 \\ 2 & 0 & 3 & 1 & 0 & 6 & -2 & -1 \\ 3 & 0 & -1 & 0 & 1 & -2 & 3 & 1 \\ \hline \end{array} \quad \begin{array}{l} \\ 7/2 \\ \\ 3/1 \end{array}$$

(a) (5 points) Write down a pivot matrix to represent the simplex pivot.

(b) (10 points) **Use the pivot matrix** to find the reduced costs and hence show that the resulting tableau is in optimal form. What is the optimal solution?

(a) $B^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Reduced cost: $c_N - c_B^T B^{-1} N$

Basic variables: x_1, x_3, x_7.

So $c_B^T = (0, 0, -1)$ and $c_B^T B^{-1} = (0, 0, -1)$.

Reduced cost for x_2:

$c_2 - c_B^T B^{-1} a_2 = 1 - (0, 0, -1)\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = 0 \geq 0$ ✓

Reduced cost for x_5:

$c_5 - c_B^T B^{-1} a_5 = 5 - (0, 0, -1)\begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix} = 3 \geq 0$ ✓

Reduced cost for x_6:

$c_6 - c_B^T B^{-1} a_6 = 1 - (0, 0, -1)\begin{pmatrix} 7 \\ -2 \\ 3 \end{pmatrix} = 4 \geq 0$ ✓

So solution is optimal after the pivot.

$$\begin{pmatrix} x_1 \\ x_3 \\ x_7 \end{pmatrix} = B^{-1} b = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}}}$$

4. (15 points) Consider the linear programming problem

$$\begin{array}{llllllll} \min & x_1 & & - & 3x_3 & + & 2x_4 & \\ \text{s.t.} & x_1 & + \; x_2 & - & 4x_3 & & & = 3 \\ & x_1 & - \; x_2 & + & 2x_3 & + & 2x_4 & = 1 \\ & & x_i & \geq & 0, & & i & = 1, \ldots, 4. \end{array}$$

Use the method of artificial variables to find an initial canonical form for this linear program. You should obtain the basic feasible solution $x_1 = 2$, $x_2 = 1$, $x_3 = x_4 = 0$. (If you have a choice for the entering column, choose the first possible column. You should only need two pivots to solve the artificial problem, once you have a canonical form for the artificial problem.)

	x_1	x_2	x_3	x_4	y_1	y_2
0	0	0	0	0	1	1
3	1	1	-4	0	1	0
1	1	-1	2	2	0	1

→

	-4	-2	0	2	-2	0	0
3/1	3	1	1	-4	0	1	0
1/1	1	1	-1	2	2	0	1

-2	0	-2	6	2	0	2
2	0	2	-6	-2	1	-1
1	1	-1	2	2	0	1

→

0	0	0	0	0	1	1
1	0	1	-3	-1	$\frac{1}{2}$	$-\frac{1}{2}$
2	1	0	-1	1	$\frac{1}{2}$	$\frac{1}{2}$

Put in original objective function

Delete y_1, y_2 columns.

0	1	0	-3	2
1	0	1	-3	-1
2	1	0	-1	1

Pivot to canonical form:

-2	0	0	-2	2
1	0	1	-3	-1
2	1	0	-1	1

(In unbounded form.

Ray: $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$)

5. (15 points) An LP of the form

$$\begin{array}{llcl} \min & c^T x & & \\ \text{subject to} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

has optimal tableau

	x_1	x_2	x_3	x_4	s_1	s_2	s_3
25	0	12	3	0	0	4	7
14	0	5	2	1	0	7	−4
6	1	3	−2	0	0	2	−1
8	0	−2	1	0	1	−2	3

How much would you be willing to pay for an additional unit of Resource 2? What is the maximum amount of this resource that you are prepared to buy at these prices? If instead someone wished to purchase the resource from you, what is the maximum you would be prepared to sell at this price?

Pay \$4 / unit.

Maximum sale ~~purchase~~: $\min \left\{ \frac{14}{7}, \frac{6}{2} \right\} = \boxed{2}$

~~Minim~~

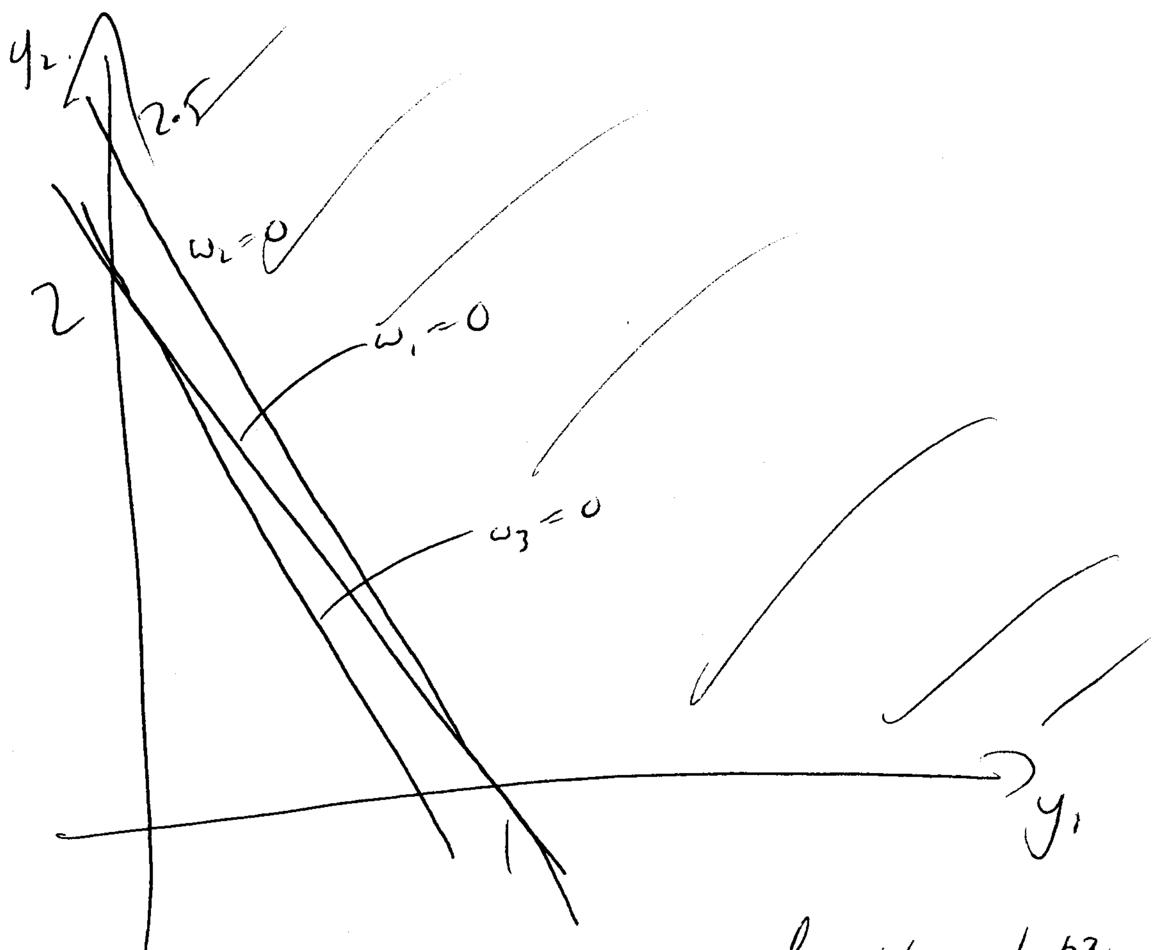
Maximum purchase ~~sale~~: $\min \left\{ \frac{8}{2} \right\} = \boxed{4}$

6. (20 points) Consider the primal-dual pair of linear programming problems

$$\begin{array}{llcrcrcrcl} \max & 2x_1 & + & 5x_2 & + & 10x_3 & & & & \\ \text{subject to} & 2x_1 & + & 5x_2 & + & 12x_3 & \leq & b_1 & & (P) \\ & x_1 & + & 2x_2 & + & 5x_3 & \leq & b_2 & & \\ & & & & & x_1, x_2, x_3 & \geq & 0 & & \end{array}$$

$$\begin{array}{llcrcl} \min & b_1y_1 & + & b_2y_2 & & \\ \text{subject to} & 2y_1 & + & y_2 & \geq & 2 \quad (D) \\ & 5y_1 & + & 2y_2 & \geq & 5 \\ & 12y_1 & + & 5y_2 & \geq & 10 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

(a) (10 points) Graph the feasible region for (D). Use this picture and complementary slackness to show that in any optimal solution to (P) we must have $x_3 = 0$.



$w_3 > 0$ in any dual feasible solution.

So $x_3 = 0$ in any optimal primal solution.

(b) (10 points) Now set $b_1 = 19$ and $b_2 = 9$. The primal-dual pair has an optimal solution with $x_1 = 7$, $x_2 = 1$, $x_3 = 0$, $y_1 = 1$, and $y_2 = 0$. It has become possible to produce an additional product x_4 that uses 3 units of resource 1 and 4 units of resource 2. What price c_4 must be charged for this product for it to be worthwhile to produce it?

Add dual constraint:

$$3y_1 + 4y_2 \geq c_4$$

Current solution is infeasible if $c_4 > 3$.

So produce if $\boxed{c_4 > 3}$