Chapter 6

Exercise Solutions

6-1. (a)

(b) The direction of most rapid improvement is objective function vector $\Delta w = (2, 3)$. (c) With no active constraints, every direction is feasible. (d) All constraints are satisfied as strict inequality. (e) $\lambda_{\max} = \min\{(12 - 7)/(4 \cdot 2 + 3 \cdot 3), (2 - 1)/(1 \cdot 3)\} = 5/17$. (g) At the boundary we must confront limits on feasible directions.

6-2. (a)

(b) The direction of most rapid improvement is the negative of the objective function vector or $\Delta w = (-9, -1)$. (c) With no active constraints, every direction is feasible. (d) All constraints are satisfied as strict inequality. (e) $\lambda_{\max} = \min\{(12 - 6)/(3 \cdot 9 + 2 \cdot 1), (13 - 6)/(2 \cdot 9 + 3 \cdot 1)\} = 6/29$. (f) $w(1) = (4/29, 81/29)$. (g) At the boundary we must confront limits on feasible directions.

6-3. (a) No, because the first point $P1$ is on the boundary. (b) Yes, because the search reaches the boundary only at optimal $P6$. (c) No, because the search visits non-optimal boundary point $P8$. (d) No, because the search visits non-optimal boundary point $P7$. (e) Yes, because the search reaches the boundary only at optimal $P6$. (f) No, because the first point $P2$ is on the boundary.

6-4. A point is interior for standard form if it is feasible in the equality constraints and all
components are positive, i.e. no nonnegativity constraint is active. (a) No, because \( x_2 = 0 \). (b) Yes, because all components are positive, \( 4(2) + 1(5) = 13 \), and \( 5(2) + 5(1) = 15 \). (c) Yes, because all components are positive, \( 4(1) + 1(9) = 13 \), and \( 5(1) + 5(2) = 15 \). (d) No, because \( 4(5) + 1(1) \neq 13 \). (e) No, because \( 4(2) + 1(1) \neq 13 \). (f) No, because \( x_1 = 0 \).

6-5. (a) The only requirements are those of equalities that \( 2 \Delta w_1 + 3 \Delta w_2 - 3 \Delta w_3 = 0 \), \( 4 \Delta w_1 - 1 \Delta w_2 + 1 \Delta w_3 = 0 \). All \( w_j \) would be positive at an interior point, so none of their nonnegativity constraints are active. (b) The only requirements are those of equalities that \( 7 \Delta w_1 + 1 \Delta w_2 - 1 \Delta w_3 = 0 \), \( 2 \Delta w_1 - 5 \Delta w_2 + 3 \Delta w_3 = 0 \). All \( w_j \) would be positive at an interior point, so none of their nonnegativity constraints are active.

6-6. (a) Here \( A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \end{pmatrix} \). Using the corresponding \( P \) of Table 6.4, \( \Delta x = Pd = (-8, -1.6, 4) \). To verify feasibility, we check that \( A \Delta x = 0 \).

(b) Here \( A = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & -2 \end{pmatrix} \). Using the corresponding \( P \) of Table 6.4, \( \Delta x = Pd = (-.2857, .8571, .4286) \). Checking feasibility, \( A \Delta x = 0 \).

6-7. (a) The direction of most rapid improvement is the negative of the objective function vector or \( d = (-14, -3, -5) \).

(b) Here \( A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \). Then

\[
P = (I - A^T(AA^T)^{-1}A) = \begin{pmatrix} 1/6 & -1/6 & 1/3 \\ -1/6 & 1/6 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}
\]

(c) \( \Delta z = Pd = (-7/2, 7/2, -7) \) (d) Checking feasibility, \( A \Delta z = 0 \), and checking improvement, \( (14, 3, 5) \cdot \Delta z < 0 \). (e) The direction is improving and feasible, and it is the closest such direction to the steepest improvement one \( d \).

6-8. (a) The direction of most rapid improvement is the objective function vector \( d = (4, -1, 7) \)
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6-11. (a) 

(c) With $X_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $c^{(3)} = cX_3$ and $A_3 = AX_3$ to produce scaled standard form:

min $4y_1 + 3y_2 + 3y_3$, s.t. $4y_1 + 5y_2 + 3y_3 = 12$, $y_1, y_2, y_3 \geq 0$

6-12. (a) 

(c) With $X_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $c^{(9)} = cX_9$ and $A_9 = AX_9$ to produce scaled standard form:

min $6y_1 + 7y_2 + 4y_3$, s.t. $1y_1 + 7y_2 + 10y_3 = 18$, $y_1, y_2, y_3 \geq 0$

6-13. (a) With $A_3 = (1, 7, 10)$, projection matrix $P_3 = (I - A_3^T(A_3A_3^T)^{-1}A_3) = \begin{pmatrix} .68 & -.40 & -.24 \\ -.40 & .50 & -.30 \\ -.24 & -.30 & .82 \end{pmatrix}$. Then $\Delta y = -P_3c^{(3)} = (-.32, 1.6, -2.24)$, and $\Delta x = X_3 \Delta y = (-.64, 1.6, -2.24)$. (b) Checking improving, $(2, 3, 5) \cdot \Delta x = -.768 < 0$, and feasible, $(2, 5, 3) \cdot \Delta x = 0.$
(c) $\lambda = 1/||\Delta x X_3^{-1}|| = .36084$

6-14. (a) With $A_9 = (1, 7, 10)$, projection matrix $P_9 = (I - A_9 A_9^T)^{-1} A_9 = \begin{pmatrix} .9933 & -.0467 & -.0667 \\ -.0467 & .6733 & -.4667 \\ -.0667 & -.4667 & .3333 \end{pmatrix}$. Then $\Delta y = P_9 c^{(0)} = (5.367, 2.567, -2.333)$, and $\Delta x = X_9 \Delta y = (5.367, 17.967, -4.667)$. (b) Checking improving $(6, 1, 2) \cdot \Delta x = 40.83 > 0$, and feasibility, $(1, 1, 5) \cdot \Delta x = 0$. (c) $\lambda = 1/||\Delta x X_9^{-1}|| = .15649$

6-15. (a) This $x^{(0)}$ is strictly positive, $(4) - (3) + 2(1) = 3$, and $(3) - (1) = 2$ as required for an interior point. (b) With $X_0 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $c^{(0)} = c X_0$ and $A_0 = A X_0$ to produce scaled standard form: $\min 40y_1 + 3y_2$, s.t. $4y_1 - 3y_2 + 2y_3 = 3$, $3y_2 - y_3 = 2$, $y_1, y_2, y_3 \geq 0$ (c) $\Delta x = -X_0 P_9 c^{(0)} = (-7.669, 7.669, 7.669)$, where $X_0$, and $c^{(0)}$ are as in part (b) and $P_9$ is the projection matrix for $A_9$ shown in Table 6.4. (d) Checking improving, $(10, 1, 0) \cdot \Delta x = 69.02 < 0$, and checking feasibility, $A \Delta x = 0$. (e) $\lambda = 1/||\Delta x X_0^{-1}|| = .12037$; $x^{(1)} = x^{(0)} + \lambda \Delta x = (3.077, 3.923, 1.923)$.

6-16. (a) This $x^{(0)}$ is strictly positive, $9(1/3) - 2(1/2) + 4(1) = 6$, and $2(1/2) - 2(1) = -1$ as required for an interior point. (b) With $X_0 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $c^{(0)} = c X_0$ and $A_0 = A X_0$ to produce scaled standard form: $\max 2y_1 + 4y_2 + 10y_3$, s.t. $3y_1 - y_2 + 4y_3 = 6$, $y_2 - 2y_3 = -1$, $y_1, y_2, y_3 \geq 0$ (c) $\Delta x = X_0 P_9 c^{(0)} = (-.6803, 3.0612, 3.0612)$ where $X_0$, and $c^{(0)}$ are as in part (b) and $P_9$ is the projection matrix for $A_9$ shown in Table 6.4. (d) Checking improving, $(6, 8, 10) \cdot \Delta x = 51.02 > 0$, and checking feasibility, $A \Delta x = 0$. (e) $\lambda = 1/||\Delta x X_0^{-1}|| = .1400$; $x^{(1)} = x^{(0)} + \lambda \Delta x = (0.2381, 0.9286, 1.4286)$.

6-17. (a) This case is unbounded because the direction has no negative components. (b) Some components of the direction are negative, so the model is not unbounded. $\lambda = 1/||\Delta x X_{11}^{-1}|| = .2425$, so that $x^{(12)} = x^{(11)} + \lambda \Delta x = (3, 1.97, 6.82)$. (c) Some components of the direction are negative, so the model is not unbounded. $\lambda = 1/||\Delta x X_{11}^{-1}|| = .1581$, so that $x^{(12)} = x^{(11)} + \lambda \Delta x = (3.949, 0.514, 9)$. (d) Some components of the direction are negative, so the model is not unbounded. $\lambda = 1/||\Delta x X_{11}^{-1}|| = 0.500$, so that $x^{(12)} = x^{(11)} + \lambda \Delta x = (3, 0, 9)$. Since a component has become $0$, the solution is optimal.

6-18. (a)
6-19. (a)

(b) \( \min 2w_1 + 5w_2 - w_3 - \mu(\ln(w_1) + \ln(w_2) + \ln(w_3)) \), s.t. \( w_1 + 6w_2 + 2w_3 = 18, w_1, w_2, w_3 \geq 0 \)
(c) At \( w^{(1)} \), original = 14 versus log barrier = -14.9. At \( w^{(2)} \), original = -8.89 versus log barrier = 61.28. We see that the log term creates a modest bonus in the middle of the feasible region, but assesses a major penalty near the boundary.
(d) With \( \mu = 100 \), the barrier optimum is \( w^* = (5.571, 0.993, 3.236) \); with \( \mu = 10 \) it is \( w^* = (2.910, 0.734, 5.342) \); and with \( \mu = 1 \) it is \( w^* = (0.391, 0.120, 8.446) \).
(e) With relatively large \( \mu \) the optimum is in the middle of the feasible region, but as \( \mu \) approaches zero, the optimum tends to an optimal solution for the original LP.

6-20. (a) Yes, because multipliers are decreasing.
(b) No, because multipliers are increasing.
(c) No, because multipliers do not always decrease.
(d) Yes, because multipliers are decreasing.

6-21. (a) Curve II, because the move for this maximize model will improve the barrier objective for a while, but drop off to \(-\infty\) as we approach the boundary.
(b) Curve IV, because the move for the minimize model will improve the barrier objective for a while, but increase to \(+\infty\) as we approach the boundary.

6-22. (a) All components are positive, \( 4(3) - 3(1) + 2(2) = 13 \), and \( 3(1) - (2) = 1 \) as required for an interior point.
(b) \( \min 4x_1 - x_2 + 2x_3 - 10(\ln(x_1) + \ln(x_2) + \ln(x_3)) \), s.t. \( 4x_1 - 3x_2 + 2x_3 = 13, 3x_2 - x_3 = 1, x_1, x_2, x_3 \geq 0 \)
(c) Here \( X_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \), so that \( A_0 = \begin{pmatrix} 12 & -3 & 4 \\ 0 & 3 & -2 \end{pmatrix} \)
and \( c^{(0)} = (12, -1, 4) \). Then
\[ \Delta x = -X_0P_0 \begin{pmatrix} 12 - 10 \\ -1 - 10 \\ 4 - 10 \end{pmatrix} = (-4.6415, 6.1887, 18.566), \]
where \( P_0 \) is the corresponding projection matrix from Table 6.4.
(d) Checking improving, the gradient of the barrier function is \( \begin{pmatrix} c_1 - \mu/x_1, c_2 - \mu/x_2, c_3 - \mu/x_3 \end{pmatrix} = (0.667, -11, -3) \text{ and } (0.667, -11, -3) \cdot \Delta x = -126.8 < 0 \). Checking feasibility, \( A\Delta x = 0 \).
(e) The maximum feasible step is \( \lambda_{\text{max}} = \min\{1/3/4.6415\} = 0.6643 \), so that \( \lambda = \min\{0.9(0.6463), 1/10\} = 0.10 \).
(f) The barrier function will decrease for a while because the direction is improving for this minimize problem, but it will increase again near the boundary.

6-23. (a) All components are positive, \( (1) + 2(5) + (3) = 14 \), and \( -2(1) + (5) = 3 \) as required for an interior point.
(b) \( \max -x_1 + 3x_2 + 8x_3 + 10(\ln(x_1) + \ln(x_2) + \ln(x_3)) \), s.t. \( x_1 + 2x_2 + 2x_3 = 14, -2x_1 + x_2 = 3, x_1, x_2, x_3 \geq 0 \)
(c) Here \( X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \), so that \( A_0 = \begin{pmatrix} 1 & 10 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix} \)
and \( c^{(0)} = (-1, 15, 24) \). Then
\[ \Delta x = X_0P_0 \begin{pmatrix} -1 + 10 \\ 15 + 10 \\ 24 + 10 \end{pmatrix} = (-9.5655, -19.131, 47.827), \]
where \( P_0 \) is the corresponding projection matrix from Table 6.4.
(d) Checking improving, the gradient of the barrier function is \( \begin{pmatrix} c_1 + \mu/x_1, c_2 + \mu/x_2, c_3 + \mu/x_3 \end{pmatrix} = (9.5, 11.33) \text{ and } (9.5, 11.33) \cdot \Delta x = 360.1 > 0 \). Checking feasibility, \( A\Delta x = 0 \).
(e) The maximum feasible step is \( \lambda_{\text{max}} = \min\{1/9.5655, 5/19.131\} = 0.1045 \), so that \( \lambda = \min\{0.9(0.1045), 1/10\} = 0.09405 \).
(f) The barrier function will increase for a while because the direction is improving for this maximize problem, but it will decrease again near the boundary.
(g) The barrier multiplier should be decreased toward 0.

6-24. (a) See Exercise 6-15(a). (b) Here \( X_0 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \), so that \( A_0 = \begin{pmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \end{pmatrix} \), and \( c^{(0)} = (40, 3, 0) \). Then

\[
\Delta x = -X_0 P_0 \begin{pmatrix} 40 - 10 \\ -1 - 10 \\ 4 - 10 \end{pmatrix} = (-16.899, 16.899, 16.899) \]

where \( P_0 \) is the corresponding projection matrix from Table 6.4. (c) Checking improving, the gradient of the barrier function is \((c_1 - \mu/x_1, c_2 - \mu/x_2, c_3 - \mu/x_3) = (7.5, -2.333, -10). \) Then \((7.5, -2.333, -10) \cdot \Delta x = -335.0 < 0 \). Checking feasibility, \( A \Delta x = 0 \). (d) The maximum feasible step is \( \lambda_{max} \min\{4/16.889\} = 0.2367. \)

(e) \( \lambda = \min\{0.9(0.2367), 1/10\} = 0.1. \) \( x^{(1)} = x^{(0)} + \lambda \Delta x = (2.310, 4.690, 2.690). \)

6-25. (a) See Exercise 6-16(a). (b) Here \( X_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \), so that \( A_0 = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & -2 \end{pmatrix} \), and \( c^{(0)} = (2, 4, 10) \). Then

\[
\Delta x = X_0 P_0 \begin{pmatrix} 2 + 10 \\ 4 + 10 \\ 10 + 10 \end{pmatrix} = (-1.633, 7.347, 7.347) \]

where \( P_0 \) is the corresponding projection matrix from Table 6.4. (c) Checking improving, the gradient of the barrier function is \((c_1 + \mu/x_1, c_2 + \mu/x_2, c_3 + \mu/x_3) = (36, 28, 20). \) Then \((36, 28, 20) \cdot \Delta x = 294.2 > 0 \). Checking feasibility, \( A \Delta x = 0 \). The maximum feasible step is \( \lambda_{max} \min\{0.3333/1.633\} = 0.2042. \) (d) \( \lambda = \min\{0.9(0.2042), 1/10\} = 0.1; \)

\( x^{(1)} = x^{(0)} + \lambda \Delta x = (.170, 1.235, 1.735). \)