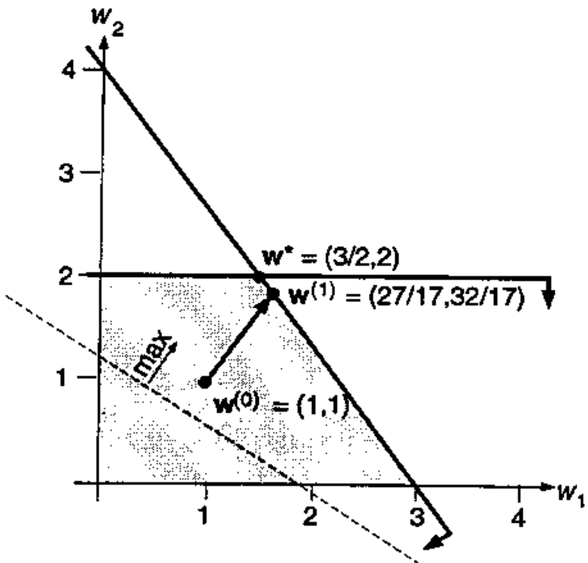
# Chapter 6

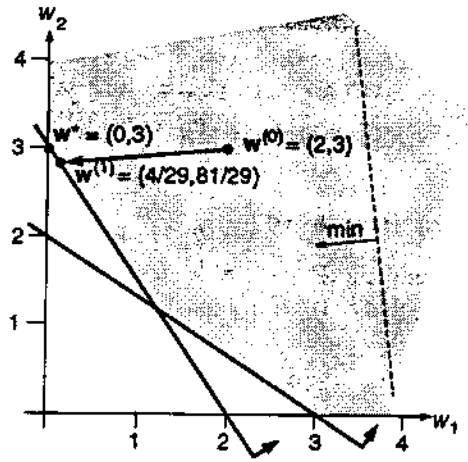
# Exercise Solutions

**6-1.**
**(a)**




**(b)** The direction of most rapid improvement is objective function vector $\Delta\mathbf{w} = (2, 3)$. **(c)** With no active constraints, every direction is feasible. **(d)** All constraints are satisfied as strict inequality. **(e)** $\lambda_{max} = \min\{(12-7)/(4 \cdot 2 + 3 \cdot 3), (2-1)/(1 \cdot 3)\} = 5/17$. **(g)** At the boundary we must confront limits on feasible directions.

**6-2.**
**(a)**




**(b)** The direction of most rapid improvement is the negative of the objective function vector or $\Delta\mathbf{w} = (-9, -1)$. **(c)** With no active constraints, every direction is feasible. **(d)** All constraints are satisfied as strict inequality. **(e)** $\lambda_{max} = \min\{(12-6)/(3 \cdot 9+2 \cdot 1), (13-6)/(2 \cdot 9+3 \cdot 1)\} = 6/29$ $\mathbf{w}^{(1)} = (4/29, 81/29)$ **(g)** At the boundary we must confront limits on feasible directions.

**6-3.** **(a)** No, because the first point $P1$ is on the boundary. **(b)** Yes, because the search reaches the boundary only at optimal $P6$. **(c)** No, because the search visits non-optimal boundary point $P8$. **(d)** No, because the search visits non-optimal boundary point $P7$. **(e)** Yes, because the search reaches the boundary only at optimal $P6$. **(f)** No, because the first point $P2$ is on the boundary.

**6-4.** A point is interior for standard form if it is feasible in the equality constraints and all components are positive, i.e. no nonnegativity constraint is active. **(a)** No, because $x_2 = 0$. **(b)** Yes, because all components are positive, $4(2)+1(5) = 13$, and $5(2)+5(1) = 15$. **(c)** Yes, because all components are positive, $4(1)+1(9) = 13$, and $5(1)+5(2) = 15$. **(d)** No, because $4(5)+1(1) \neq 13$. **(e)** No, because $4(2)+1(1) \neq 13$. **(f)** No, because $x_1 = 0$.

**6-5.** **(a)** The only requirements are those of equalities that $2\,\Delta w_1 + 3\,\Delta w_2 - 3\,\Delta w_3 = 0$, $4\,\Delta w_1 - 1\,\Delta w_2 + 1\,\Delta w_3 = 0$. All $w_j$ would be positive at an interior point, so none of their nonnegativity constraints are active. **(b)** The only requirements are those of equalities that $7\,\Delta w_1 + 1\,\Delta w_2 - 1\,\Delta w_3 = 0$, $2\,\Delta w_1 - 5\,\Delta w_2 + 3\,\Delta w_3 = 0$ All $w_j$ would be positive at an interior point, so none of their nonnegativity constraints are active.

**6-6.** **(a)** Here $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$. Using the corresponding **P** of Table 6.4, $\Delta\mathbf{x} = \mathbf{Pd} = (-.8, -1.6, 4)$. To verify feasibility, we check that $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$.
**(b)** Here $\mathbf{A} = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & -2 \end{pmatrix}$. Using the corresponding **P** of Table 6.4, $\Delta\mathbf{x} = \mathbf{Pd} = (-.2857, .8571, .4286)$. Checking feasibility, $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$.

**6-7.** **(a)** The direction of most rapid improvement is the negative of the objective function vector or $\mathbf{d} = (-14, -3, -5)$.
**(b)** Here $\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$. Then

$$\mathbf{P} = (\mathbf{I} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}) = \begin{pmatrix} 1/6 & -1/6 & 1/3 \\ -1/6 & 1/6 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}$$

**(c)** $\Delta\mathbf{z} = \mathbf{Pd} = (-7/2, 7/2, -7)$ **(d)** Checking feasibility, $\mathbf{A}\,\Delta\mathbf{z} = \mathbf{0}$, and checking improvement, $(14,3,5)\cdot\Delta\mathbf{z} < 0$. **(e)** The direction is improving and feasible, and it is the closest such direction to the steepest improvement one **d**.

**6-8.** **(a)** The direction of most rapid improvement is the objective function vector $\mathbf{d} = (4, -1, 7)$
**(b)** Here $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$. Then $\mathbf{P} =$

$$(\mathbf{I} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}) = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

**(c)** $\Delta\mathbf{z} = \mathbf{Pd} = (-3/2, 0, 3/2)$ **(d)** Checking feasibility, $\mathbf{A}\,\Delta\mathbf{z} = \mathbf{0}$, and checking improvement, $(14,3,5)\cdot\Delta\mathbf{z} > 0$. **(e)** The direction is improving and feasible, and it is the closest such direction to the steepest improvement one **d**.

**6-9.** In each case $\mathbf{y} = \mathbf{X}_7^{-1}\mathbf{x}$ where

$$\mathbf{X}_7 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

**(a)** For $\mathbf{x} = (1,1,1,1)$ this gives $(1/2, 1/5, 1, 1/9)$.
**(b)** For $\mathbf{x} = (4,2,3,5)$ this gives $(2, 2/5, 3, 5/9)$.
**(c)** For $\mathbf{x} = (3,5,1,6)$ this gives $(3/2, 1, 1, 2/3)$.
**(d)** For $\mathbf{x} = (2,5,1,9)$ this gives $(1,1,1,1)$.

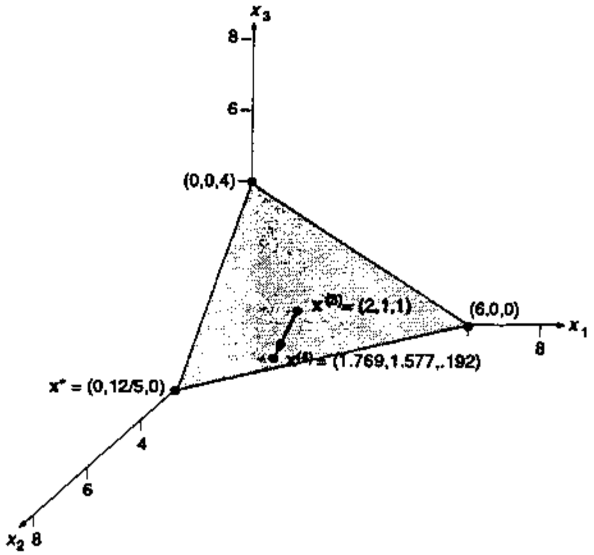
**6-10.** In each case $\mathbf{x} = \mathbf{X}_7\mathbf{y}$ where

$$\mathbf{X}_7 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

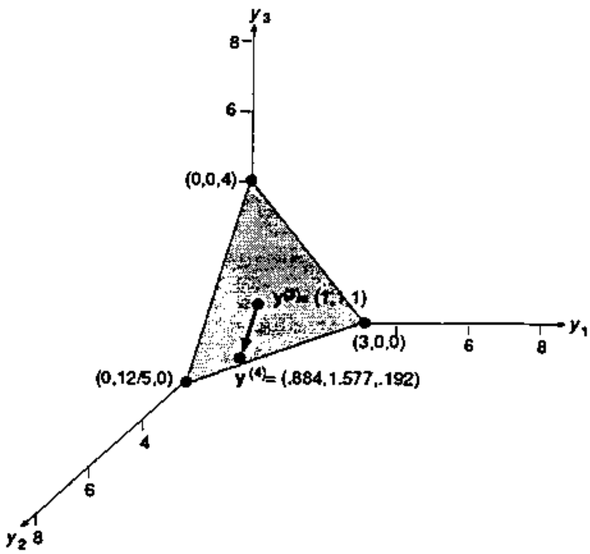
**(a)** For $\mathbf{y} = (1,1,1,1)$ this gives $(2,5,1,9)$.
**(b)** For $\mathbf{y} = (4,2,3,5)$ this gives $(8,10,3,45)$.
**(c)** For $\mathbf{y} = (3,5,1,6)$ this gives $(6,25,1,54)$.
**(d)** For $\mathbf{y} = (2,5,1,9)$ this gives $(4,25,1,81)$.

**6-11.**

**(a)**



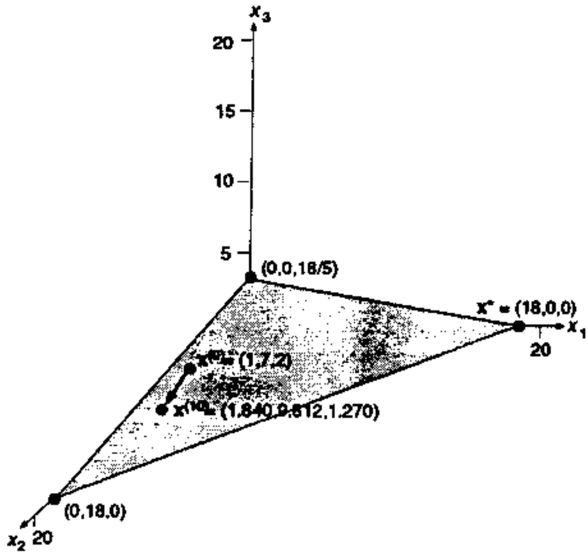

**(b)**



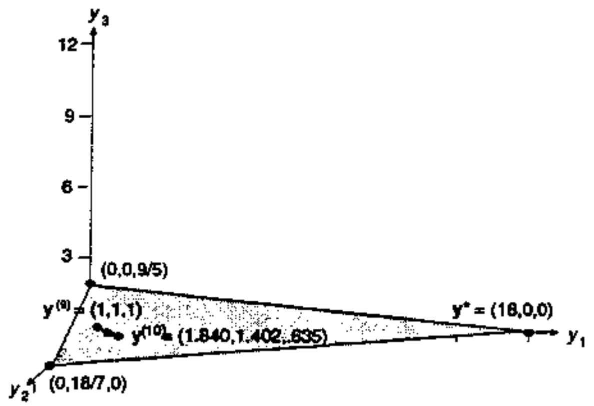

**(c)** With $\mathbf{X}_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{c}^{(3)} = \mathbf{c}\mathbf{X}_3$ and $\mathbf{A}_3 = \mathbf{A}\mathbf{X}_3$ to produce scaled standard form: min $4y_1 + 3y_2 + 5y_3$, s.t. $4y_1 + 5y_2 + 3y_3 = 12$, $y_1, y_2, y_3 \geq 0$

**6-12.**

**(a)**




**(b)**




**(c)** With $\mathbf{X}_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\mathbf{c}^{(9)} = \mathbf{c}\mathbf{X}_9$ and $\mathbf{A}_9 = \mathbf{A}\mathbf{X}_9$ to produce scaled standard form: min $6y_1 + 7y_2 + 4y_3$, s.t. $1y_1 + 7y_2 + 10y_3 = 18$, $y_1, y_2, y_3 \geq 0$

**6-13.** **(a)** With $\mathbf{A}_3 = (1, 7, 10)$, projection matrix $\mathbf{P}_3 = (\mathbf{I} - \mathbf{A}_3{}^T(\mathbf{A}_3\mathbf{A}_3{}^T)^{-1}\mathbf{A}_3) = \begin{pmatrix} .68 & -.40 & -.24 \\ -.40 & .50 & -.30 \\ -.24 & -.30 & .82 \end{pmatrix}$. Then $\Delta\mathbf{y} = -\mathbf{P}_3\mathbf{c}^{(3)} = (-.32, 1.6, -2.24)$, and $\Delta\mathbf{x} = \mathbf{X}_3\Delta\mathbf{y} = (-.64, 1.6, -2.24)$. **(b)** Checking improving, $(2,3,5)\cdot\Delta\mathbf{x} = -7.68 < 0$, and feasible, $(2,5,3)\cdot\Delta\mathbf{x} = 0$.

(c) $\lambda = 1/\|\Delta x X_3^{-1}\| = .36084$

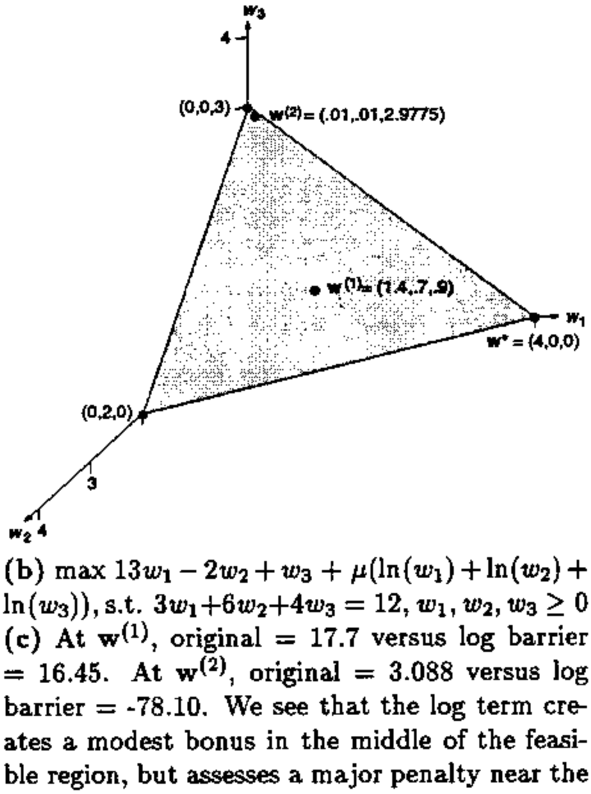
**6-14.** (a) With $\mathbf{A}_9 = (1, 7, 10)$, projection matrix $\mathbf{P}_9 = (\mathbf{I} - \mathbf{A_9}^T(\mathbf{A}_9\mathbf{A_9}^T)^{-1}\mathbf{A}_9) = \begin{pmatrix} .9933 & -.0467 & -.0667 \\ -.0467 & .6733 & -.4667 \\ -.0667 & -.4667 & .3333 \end{pmatrix}$. Then $\Delta\mathbf{y} = \mathbf{P}_9\mathbf{c}^{(9)} = (5.367, 2.567, -2.333)$, and $\Delta\mathbf{x} = \mathbf{X}_9\,\Delta\mathbf{y} = (5.367, 17.967, -4.667)$. (b) Checking improving $(6, 1, 2)\cdot\Delta\mathbf{x} = 40.83 > 0$, and feasibility, $(1, 1, 5)\cdot\Delta\mathbf{x} = 0$. (c) $\lambda = 1/\|\Delta\mathbf{x}\mathbf{X}_9^{-1}\| = .15649$

**6-15.** (a) This $\mathbf{x}^{(0)}$ is strictly positive, $(4)-(3)+2(1) = 3$, and $(3)-(1) = 2$ as required for an interior point. (b) With $\mathbf{X}_0 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{c}^{(0)} = \mathbf{c}\mathbf{X}_0$ and $\mathbf{A}_0 = \mathbf{A}\mathbf{X}_0$ to produce scaled standard form: min $40y_1 + 3y_2$, s.t. $4y_1 - 3y_2 + 2y_3 = 3$, $3y_2 - y_3 = 2$, $y_1, y_2, y_3 \geq 0$ (c) $\Delta\mathbf{x} = -\mathbf{X}_0\mathbf{P}_0\mathbf{c}^{(0)} = (-7.669, 7.669, 7.669)$, where $\mathbf{X}_0$, and $\mathbf{c}^{(0)}$ are as in part (b) and $\mathbf{P}_0$ is the projection matrix for $\mathbf{A}_0$ shown in Table 6.4. (d) Checking improving, $(10, 1, 0)\cdot\Delta\mathbf{x} = -69.02 < 0$, and checking feasible, $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$. (e) $\lambda = 1/\|\Delta\mathbf{x}\mathbf{X}_0^{-1}\| = .12037$; $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \lambda\,\Delta\mathbf{x} = (3.077, 3.923, 1.923)$.

**6-16.** (a) This $\mathbf{x}^{(0)}$ is strictly positive, $9(1/3) - 2(1/2) + 4(1) = 6$, and $2(1/2) - 2(1) = -1$ as required for an interior point. (b) With $\mathbf{X}_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{c}^{(0)} = \mathbf{c}\mathbf{X}_0$ and $\mathbf{A}_0 = \mathbf{A}\mathbf{X}_0$ to produce scaled standard form: max $2y_1 + 4y_2 + 10y_3$, s.t. $3y_1 - 1y_2 + 4y_3 = 6$, $y_2 - 2y_3 = -1$, $y_1, y_2, y_3 \geq 0$ (c) $\Delta\mathbf{x} = \mathbf{X}_0\mathbf{P}_0\mathbf{c}^{(0)} = (-.6803, 3.0612, 3.0612)$ where $\mathbf{X}_0$, and $\mathbf{c}^{(0)}$ are as in part (b) and $\mathbf{P}_0$ is the projection matrix for $\mathbf{A}_0$ shown in Table 6.4. (d) Checking improving, $(6, 8, 10)\cdot\Delta\mathbf{x} = 51.02 > 0$, and checking feasibility, $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$ (e) $\lambda = 1/\|\Delta\mathbf{x}\mathbf{X}_0^{-1}\| = .1400$; $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \lambda\,\Delta\mathbf{x} = (0.2381, 0.9286, 1.4286)$.

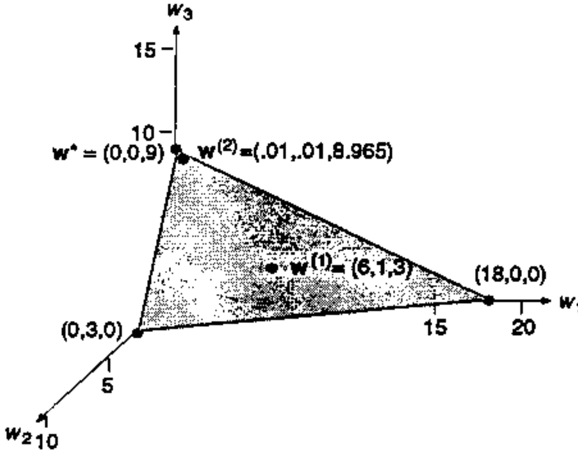
**6-17.** (a) This case is unbounded because the direction has no negative components. (b) Some components of the direction are negative, so the model is not unbounded. $\lambda = 1/\|\Delta\mathbf{x}\mathbf{X}_{11}^{-1}\| = .2425$, so that $\mathbf{x}^{(12)} = \mathbf{x}^{(11)} + \lambda\,\Delta\mathbf{x} = (3, 1.97, 6.82)$. (c) Some components of the direction are negative, so the model is not unbounded. $\lambda = 1/\|\Delta\mathbf{x}\mathbf{X}_{11}^{-1}\| = .1581$, so that $\mathbf{x}^{(12)} = \mathbf{x}^{(11)} + \lambda\,\Delta\mathbf{x} = (3.949, 0.514, 9)$. (d) Some components of the direction are negative, so the model is not unbounded. $\lambda = 1/\|\Delta\mathbf{x}\mathbf{X}_{11}^{-1}\| = 0.500$, so that $\mathbf{x}^{(12)} = \mathbf{x}^{(11)} + \lambda\,\Delta\mathbf{x} = (3, 0, 9)$. Since a component has become $=0$, the solution is optimal.

**6-18.**
(a)




(b) max $13w_1 - 2w_2 + w_3 + \mu(\ln(w_1) + \ln(w_2) + \ln(w_3))$, s.t. $3w_1 + 6w_2 + 4w_3 = 12$, $w_1, w_2, w_3 \geq 0$ (c) At $\mathbf{w}^{(1)}$, original $= 17.7$ versus log barrier $= 16.45$. At $\mathbf{w}^{(2)}$, original $= 3.088$ versus log barrier $= -78.10$. We see that the log term creates a modest bonus in the middle of the feasible region, but assesses a major penalty near the boundary. (d) With $\mu = 100$ the barrier optimum is $\mathbf{w}^* = (1.497, 0.619, 0.949)$; with $\mu = 10$ it is $\mathbf{w}^* = (2.799, 0.285, 0.474)$; and with $\mu = 1$ it is $\mathbf{w}^* = (3.850, 0.035, 0.060)$. (e) With relatively large $\mu$ the optimum is in the middle of the feasible region, but as $\mu$ approaches zero, the optimum tends to an optimal solution for the original LP.

**6-19.**
(a)




(b) min $2w_1 + 5w_2 - w_3$ - $\mu(\ln(w_1) + \ln(w_2) + \ln(w_3))$, s.t. $w_1+6w_2+2w_3 = 18$, $w_1, w_2, w_3 \geq 0$ (c) At $\mathbf{w}^{(1)}$, original = 14 versus log barrier = -14.9. At $\mathbf{w}^{(2)}$, original = -8.89 versus log barrier = 61.28. We see that the log term creates a modest bonus in the middle of the feasible region, but assesses a major penalty near the boundary. (d) With $\mu = 100$, the barrier optimum is $\mathbf{w}^* = (5.571, 0.993, 3.236)$; with $\mu = 10$ it is $\mathbf{w}^* = (2.910, 0.734, 5.342)$; and with $\mu = 1$ it is $\mathbf{w}^* = (0.391, 0.120, 8.446)$ (e) With relatively large $\mu$ the optimum is in the middle of the feasible region, but as $\mu$ approaches zero, the optimum tends to an optimal solution for the original LP.

**6-20.** (a) Yes, because multipliers are decreasing. (b) No, because multipliers are increasing. (c) No, because multipliers do not always decrease. (d) Yes, because multipliers are decreasing.

**6-21.** (a) Curve II, because the move for this maximize model will improve the barrier objective for a while, but drop off to $-\infty$ as we approach the boundary. (b) Curve IV, because the move for the minimize model will improve the barrier objective for a while, but increase to $+\infty$ as we approach the boundary.

**6-22.** (a) All components are positive, $4(3) - 3(1) + 2(2) = 13$, and $3(1) - (2) = 1$ as required for an interior point. (b) min $4x_1 - x_2 + 2x_3$ - $10(\ln(x_1)+\ln(x_2)+\ln(x_3))$, s.t. $4x_1-3x_2+2x_3 = 13$, $3x_2 - x_3 = 1$, $x_1, x_2, x_3 \geq 0$ (c) Here $\mathbf{X}_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, so that $\mathbf{A}_0 = \begin{pmatrix} 12 & -3 & 4 \\ 0 & 3 & -2 \end{pmatrix}$ and $\mathbf{c}^{(0)} = (12, -1, 4)$. Then

$$\Delta\mathbf{x} = -\mathbf{X}_0\mathbf{P}_0 \begin{pmatrix} 12 - 10 \\ -1 - 10 \\ 4 - 10 \end{pmatrix}$$

$= (-4.6415, 6.1887, 18.566)$, where $\mathbf{P}_0$ is the corresponding projection matrix from Table 6.4. (d) Checking improving, the gradient of the barrier function is $(c_1 - \mu/x_1, c_2 - \mu/x_2, c_3 - \mu/x_3)$ $= (0.667, -11, -3)$, and $(0.667, -11, -3) \cdot \Delta\mathbf{x} =$ -126.8 $< 0$. Checking feasibility, $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$. (e) The maximum feasible step is $\lambda_{max}$ $= \min\{3/4.6415\} = 0.6463$, so that $\lambda = \min\{0.9(.6463), 1/10\} = 0.10$. (f) The barrier function will decrease for a while because the direction is improving for this minimize problem, but it will increase again near the boundary. (g) The barrier multiplier should be decreased toward 0.

**6-23.** (a) All components are positive, $(1) + 2(5) + (3) = 14$, and $-2(1) + (5) = 3$ as required for an interior point. (b) max $-x_1 + 3x_2 + 8x_3 + 10(\ln(x_1) + \ln(x_2) + \ln(x_3))$, s.t. $x_1 + 2x_2 + x_3 = 14$, $-2x_1 + x_2 = 3$, $x_1, x_2, x_3 \geq 0$ (c) Here $\mathbf{X}_0$ $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, so that $\mathbf{A}_0 = \begin{pmatrix} 1 & 10 & 3 \\ -2 & 5 & 0 \end{pmatrix}$ and $\mathbf{c}^{(0)} = (-1, 15, 24)$. Then

$$\Delta\mathbf{x} = \mathbf{X}_0\mathbf{P}_0 \begin{pmatrix} -1 + 10 \\ 15 + 10 \\ 24 + 10 \end{pmatrix}$$

$= (-9.5655, -19.131, 47.827)$, where $\mathbf{P}_0$ is the corresponding projection matrix from Table 6.4. (d) Checking improving, the gradient of the barrier function is $(c_1 + \mu/x_1, c_2 + \mu/x_2, c_3 + \mu/x_3) = (9, 5, 11.33)$ and $(9, 5, 11.33) \cdot \Delta\mathbf{x} = 360.1 > 0$. Checking feasibility, $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$. (e) The maximum feasible step is $\lambda_{max}$ $\min\{1/9.5655, 5/19.131\} = 0.1045$, so that $\lambda = \min\{0.9(.1045), 1/10\} = 0.09405$. (f) The barrier function will increase for a while because the direction is improving for this maximize problem, but it will decrease again near the boundary.

**(g)** The barrier multiplier should be decreased toward 0.

**6-24.** **(a)** See Exercise 6-15(a). **(b)** Here $\mathbf{X}_0 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so that $\mathbf{A}_0 = \begin{pmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \end{pmatrix}$, and $\mathbf{c}^{(0)} = (40, 3, 0)$. Then

$$\Delta\mathbf{x} = -\mathbf{X}_0\mathbf{P}_0 \begin{pmatrix} 40 - 10 \\ -1 - 10 \\ 4 - 10 \end{pmatrix}$$

$= (-16.899, 16.899, 16.899)$ where $\mathbf{P}_0$ is the corresponding projection matrix from Table 6.4. **(c)** Checking improving, the gradient of the barrier function is $(c_1 - \mu/x_1, c_2 - \mu/x_2, c_3 - \mu/x_3) = (7.5, -2.333, -10)$. Then $(7.5, -2.333, -10) \cdot \Delta\mathbf{x}$= -335.0 $< 0$. Checking feasibility, $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$. **(d)** The maximum feasible step is $\lambda_{max}$ $\min\{4/16.889\} = 0.2367$. **(e)** $\lambda = \min\{0.9(.2367), 1/10\} = 0.1$. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \lambda\,\Delta\mathbf{x} = (2.310, 4.690, 2.690)$.

**6-25.** **(a)** See Exercise 6-16(a). **(b)** Here $\mathbf{X}_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so that $\mathbf{A}_0 = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & -2 \end{pmatrix}$, and $\mathbf{c}^{(0)} = (2, 4, 10)$. Then

$$\Delta\mathbf{x} = \mathbf{X}_0\mathbf{P}_0 \begin{pmatrix} 2 + 10 \\ 4 + 10 \\ 10 + 10 \end{pmatrix}$$

$= (-1.633, 7.347, 7.347)$ where $\mathbf{P}_0$ is the corresponding projection matrix from Table 6.4. **(c)** Checking improving, the gradient of the barrier function is $(c_1 + \mu/x_1, c_2 + \mu/x_2, c_3 + \mu/x_3)$ $= (36, 28, 20)$. Then $(36, 28, 20) \cdot \Delta\mathbf{x} = 294.2$ $> 0$. Checking feasibility, $\mathbf{A}\,\Delta\mathbf{x} = \mathbf{0}$. The maximum feasible step is $\lambda_{max}$ $\min\{.3333/1.633\} =$ 0.2042. **(d)** $\lambda = \min\{0.9(.2042), 1/10\} = 0.1$; $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \lambda\,\Delta\mathbf{x} = (.170, 1.235, 1.735)$.