Chapter 3

Exercise Solutions

3-1. (a) $\mathbf{x}^{(1)}$ is feasible and only a local max because all constraints are satisfied and no nearby point has better objective value, but there are better such as $(1,3)$. $\mathbf{x}^{(2)}$ is infeasible and thus no sort of optimum because it violates constraint $x_2 \geq 0$. $\mathbf{x}^{(3)}$ is feasible because it satisfies all constraints, but no sort of optimum because it can be improved in the neighborhood. $\mathbf{x}^{(4)}$ is feasible and both a local and a global max because it satisfies all constraints and has better objective value than any other point in the feasible region. (b) $\mathbf{x}^{(1)}$ is infeasible and thus no sort of optimum because it violates constraint $x_1 + x_2 \leq 4$. $\mathbf{x}^{(2)}$ is feasible and only a local min because it satisfies all constraints and cannot be improved in the neighborhood, but there are better feasible points such as (2,1). $\mathbf{x}^{(3)}$ is feasible because it satisfies all constraints, but no sort of optimum can be improved in the neighborhood. $\mathbf{x}^{(4)}$ is feasible and both a local and a global min because it satisfies all constraints and has better objective value than any other point in the feasible region.

3-2. (a) $\mathbf{y}^{(1)} = (2,0,5)+2(3,-1,0) = (8,-2,5)$, $\mathbf{y}^{(2)} = (8,-2,5)+5(-1,2,1) = (3,8,10)$, $\mathbf{y}^{(3)} = (3,8,10)+1/2(0,6,0) = (3,11,10)$
(b) $\mathbf{y}^{(1)} = (2,0,5)+3(-1,4,1) = (-1,12,8)$, $\mathbf{y}^{(2)} = (-1,12,8)+1/3(0,3,-6) = (-1,13,6)$, $\mathbf{y}^{(3)} = (-1,13,6)+1(1,1,1) = (0,14,7)$

3-3. (a) $\Delta\mathbf{w}^{(1)} = (4,-1,7)-(0,1,1) = (4,-2,6)$,
$\Delta\mathbf{w}^{(2)} = (4,-3,19)-(4,-1,7) = (0,-2,8)$,
$\Delta\mathbf{w}^{(3)} = (3,-3,22)-(4,-3,19) = (-1,0,3)$
(b) $\Delta\mathbf{w}^{(1)} = (2,2,7)-(9,0,3) = (-7,2,4)$,
$\Delta\mathbf{w}^{(2)} = (3,6,7)-(2,2,7) = (1,4,0)$,
$\Delta\mathbf{w}^{(3)} = (3,9,8)-(3,6,7) = (0,3,1)$

3-4. (a) Nonimproving because the objective value worsens in the neighborhood along this direction. (b) Nonimproving because the objective decreases in the neighborhood along this direction. (c) Improving because the objective improves in the neighborhood along this direction. (d) Improving because the objective improves in the neighborhood along this direction. (e) Nonimproving because the objective value worsens in the neighborhood along this direction. (f) Nonimproving because the objective value worsens in the neighborhood along this direction.

3-5. (a) Feasible because movement along this direction retains feasibility in the neighborhood. (b) Infeasible because any movement along this direction produces infeasibility. (c) Feasible because movement along this direction retains feasibility in the neighborhood. (d) Feasible because movement along this direction retains feasibility in the neighborhood. (e) Infeasible because any movement along this direction produces infeasibility. (f) Feasible because movement along this direction retains feasibility in the neighborhood.

3-6. (a) $(4-1\lambda)-2(0+3\lambda)+3(6-2\lambda) \leq 25$ is satisfied for all positive λ; $(4-1\lambda) \geq 0$ is satisfied for $\lambda \leq 4$; $(0+3\lambda) \geq 0$ is satisfied for all positive λ; $(6-3\lambda) \geq 0$ is satisfied for $\lambda \leq 3$. Thus the proper step is $\lambda = \min\{4,3\} = 3$. The model is not unbounded because this λ is finite.
(b) $(9-3\lambda)-2(4-3\lambda)+3(6+9\lambda) \leq 25$ is satisfied for $\lambda \leq 1/5$; $(9-3\lambda) \geq 0$ is satisfied for $\lambda \leq 3$; $(4-3\lambda) \geq 0$ is satisfied for $\lambda \leq 4/3$; $(6+9\lambda) \geq 0$ is satisfied for all positive λ. Thus the proper step is $\lambda = \min\{1/5,3,4/3\} = 1/5$. The model is not unbounded because this λ is

finite.

(c) $(0+1\lambda)-2(0+3\lambda)+3(4+1\lambda) \leq 25$ is satisfied for all positive λ; $(0+1\lambda) \geq 0$ is satisfied for all positive λ; $(0+3\lambda) \geq 0$ is satisfied for all positive λ; $(4+1\lambda) \geq 0$ is satisfied for all positive λ. Thus $\lambda = +\infty$, and the model is unbounded.

(d) $(16-4\lambda) - 2(2+0\lambda) + 3(1+3\lambda) \leq 25$ is satisfied for $\lambda \leq 2$; $(16-4\lambda) \geq 0$ is satisfied for $\lambda \leq 4$; $(2+0\lambda) \geq 0$ is satisfied for all positive λ; $(1+3\lambda) \geq 0$ is satisfied for all positive λ. Thus the proper step is $\lambda = \min\{2,4\} = 2$. The model is not unbounded because this λ is finite.

3-7. **(a)** $\nabla f(2,-3,4,0,6) = (4,0,-2,0,1)$ and $\nabla f \cdot \Delta\mathbf{y} = (4,0,-2,0,1)\cdot(2,-3,4,0,6) = 6$, so improving for a maximize.

(b) $\nabla f(2,0,8,11,-3) = (0,-1,8,6,0)$ and $\nabla f \cdot \Delta\mathbf{y} = (0,-1,8,6,0)\cdot(2,0,3,-1,0) = 18 \not< 0$, so nonimproving for a minimize.

(c) $\nabla f(3,-1) = (-1+2(3), 3+4)$ and $\nabla f \cdot \Delta\mathbf{y} = (5,7)\cdot(-7,5) = 0$, so more information is needed.

(d) $\nabla f(-1,1) = (-1+2(3), 3+4)$ and $\nabla f \cdot \Delta\mathbf{y} = (5,7)\cdot(-1,1) = 2 \not< 0$, so nonimproving for a minimize.

(e) $\nabla f(4,1) = (2(4-5), 2(1+1))$ and $\nabla f \cdot \Delta\mathbf{y} = (-2,4)\cdot(-1,2) = 12>0$, so improving for a maximize.

(f) $\nabla f(2,3) = (2(2-5), 2(3+1))$ and $\nabla f \cdot \Delta\mathbf{y} = (-6,8)\cdot(8,-6) = 0$, so more information is needed.

3-8. **(a)** $\Delta\mathbf{w} = \nabla f(2,0,5,1) = (3,-2,0,1)$
(b) $\Delta\mathbf{w} = -\nabla f(1,1,7,-1) = (-4,0,-1,5)$
(c) $\Delta\mathbf{w} = -\nabla f(3,2) = -(2(3+2)-2,-3) = (-8,3)$
(d) $\Delta\mathbf{w} = \nabla f(2,5) = (3(2)^2-4,6) = (8,6)$

3-9. **(a)** $(4-2)^2 + (0-1)^2 = 5<10$ so [i] is inactive; $2(4)-(0) = 8$ so [ii] is active; $(4)>0$ so [iii] is inactive; $(0) = 0$ so [iv] is active. **(b)** $(5-2)^2 + (2-1)^2 = 10$ so [i] is active; $2(5)-(2) = 8$ so [ii] is active; $(5)>0$ so [iii] inactive; $(2)>0$ so [iv] inactive.

3-10. **(a)** Active constraints are $3y_1 - 2y_2 + 8y_3 = 14$ and $y_2 \geq 0$. For these $\mathbf{a}^{(1)} \cdot \Delta\mathbf{y} = 3,-2,8)\cdot(0,4,1) = 0$ and $\mathbf{a}^{(4)} \cdot \Delta\mathbf{y} = 0,1,0)\cdot(0,4,1) = 4 \geq 0$ as required for feasibility.

(b) Active constraints are $3y_1 - 2y_2 + 8y_3 = 14$ and $y_2 \geq 0$. For the second, $\mathbf{a}^{(4)} \cdot \Delta\mathbf{y} = 0,1,0)\cdot(0,-4,1) = -4 \not\geq 0$ as required so infeasible.

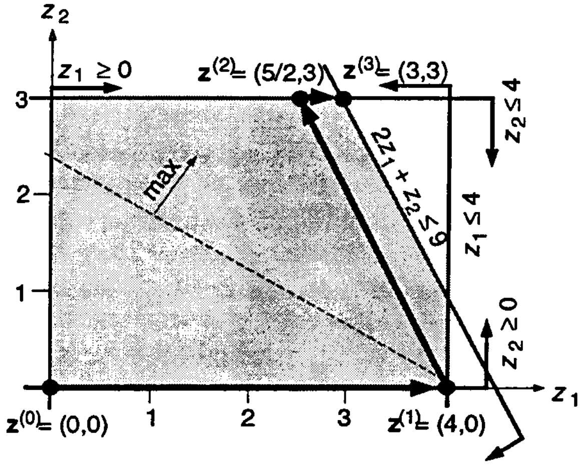
(c) Active constraints are $3y_1 - 2y_2 + 8y_3 = 14$ and $y_1 \geq 0$. In the first, $\mathbf{a}^{(1)} \cdot \Delta\mathbf{y} = (3,-2,8)\cdot(2,0,1) = 14 \neq 0$ as required so infeasible.

(d) Active constraints are $3y_1 - 2y_2 + 8y_3 = 14$ and $y_1 \geq 0$. For these $\mathbf{a}^{(1)} \cdot \Delta\mathbf{y} = (3,-2,8)\cdot(0,8,-2) = 0$ and $\mathbf{a}^{(3)} \cdot \Delta\mathbf{y} = (1,0,0)\cdot(0,8,-2) = 0 \geq 0$ as required for feasibility.

3-11. **(a)** Active constraints are $2w_1 + 3w_3 = 18$, $1w_1 + 1w_2 + 2w_3 = 14$, $w_1 \geq 0$. Thus conditions are $2\,\Delta w_1 + 3\,\Delta w_3 = 0$, $1\,\Delta w_1 + 1\,\Delta w_2 + 2\,\Delta w_3 = 0$, $\Delta w_1 \geq 0$ **(b)** Active constraints are $2w_1 + 3w_3 = 18$, $1w_1 + 1w_2 + 2w_3 = 14$, $w_3 \geq 0$. Thus conditions are $2\,\Delta w_1 + 3\,\Delta w_3 = 0$, $1\,\Delta w_1 + 1\,\Delta w_2 + 2\,\Delta w_3 = 0$, $\Delta w_3 \geq 0$ **(c)** Active constraints are $1w_1 + 1w_2 = 10$ and $2w_1 - 1w_2 \geq 8$. Thus conditions are $1\,\Delta w_1 + 1\,\Delta w_2 = 0$, $2\,\Delta w_1 - 1\,\Delta w_2 \geq 0$ **(d)** Active constraints are $1w_1 + 1w_2 = 10$ and $1w_1 - 8w_2 \leq 1$. Thus conditions are $1\,\Delta w_1 + 1\,\Delta w_2 = 0$, $1\,\Delta w_1 - 8\,\Delta w_2 \leq 0$

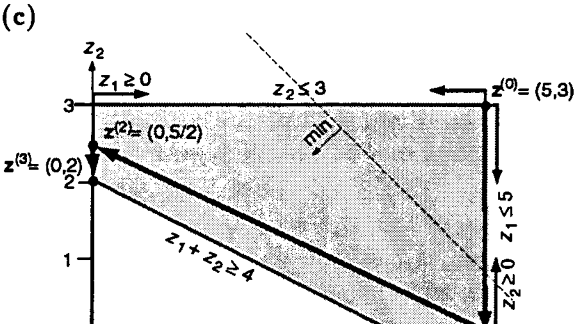
3-12. **(a)** $\nabla f \cdot \Delta\mathbf{z}^{(1)} = (4,7)\cdot(2,0) = 8>0$ and $\nabla f \cdot \Delta\mathbf{z}^{(2)} = (4,7)\cdot(-2,4) = 20>0$ as required for improving directions in a maximize.
(b) With both directions improving at all $\mathbf{z}$, the only considerations are when directions are feasible. At $\mathbf{z}^{(0)} = (0,0)$, only $\Delta\mathbf{z}^{(1)}$ is feasible because $\Delta\mathbf{z}^{(2)}$ has $\mathbf{a} \cdot \Delta\mathbf{z}^{(2)} = (1,0)\cdot(-2,4) = -2 \not\geq 0$ for active $z_1 \geq 0$. A maximum feasible step follows $\Delta\mathbf{z}^{(1)}$ for $\lambda = 2$ to $\mathbf{z}^{(1)} = (4,0)$. At $\mathbf{z}^{(1)} = (4,0)$, further pursuit of $\Delta\mathbf{z}^{(1)}$ is infeasible, but $\Delta\mathbf{z}^{(2)} = (-2,4)$ is feasible. A maximum feasible step follows $\Delta\mathbf{z}^{(2)}$ for $\lambda = 3/4$ to $\mathbf{z}^{(2)} = (5/2,3)$. At $\mathbf{z}^{(2)} = (5/2,3)$, further pursuit of $\Delta\mathbf{z}^{(2)}$ is infeasible, but $\Delta\mathbf{z}^{(1)} = (2,0)$ is again feasible. A maximum feasible step follows $\Delta\mathbf{z}^{(1)}$ for $\lambda = 1/4$ to $\mathbf{z}^{(3)} = (3,3)$. At $\mathbf{z}^{(3)}$, both directions are infeasible, and the search terminates.

(c)



3-13. **(a)** $\nabla f \cdot \Delta\mathbf{z}^{(1)} = (1,1)\cdot(0,-5) = -5<0$ and $\nabla f \cdot \Delta\mathbf{z}^{(2)} = (1,1)\cdot(-2,1) = -2<0$ as required for a minimize. **(b)** With both directions improving at all $\mathbf{z}$, the only considerations are when directions are feasible. At $\mathbf{z}^{(0)} = (5,3)$, only $\Delta\mathbf{z}^{(1)}$ is feasible because $\Delta\mathbf{z}^{(2)}$ has $\mathbf{a}\cdot\Delta\mathbf{z}^{(2)} = (0,1)\cdot(-2,1) = 1 \not\leq 0$ for active $z_2 \leq 3$. A maximum feasible step follows $\Delta\mathbf{z}^{(1)}$ for $\lambda = 3/5$ to $\mathbf{z}^{(1)} = (5,0)$. At $\mathbf{z}^{(1)} = (5,0)$, further pursuit of $\Delta\mathbf{z}^{(1)}$ is infeasible, but $\Delta\mathbf{z}^{(2)} = (-2,1)$ is now feasible. A maximum feasible step follows $\Delta\mathbf{z}^{(2)}$ for $\lambda = 5/2$ to $\mathbf{z}^{(2)} = (0,5/2)$. At $\mathbf{z}^{(2)} = (0,5/2)$, further pursuit of $\Delta\mathbf{z}^{(2)}$ is infeasible, but $\Delta\mathbf{z}^{(1)} = (0,-5$ is again feasible. A maximum feasible step follows $\Delta\mathbf{z}^{(1)}$ for $\lambda = 1/10$ to $\mathbf{z}^{(3)} = (0,2)$. At $\mathbf{z}^{(3)}$, both directions are infeasible, and the search terminates.

(c)

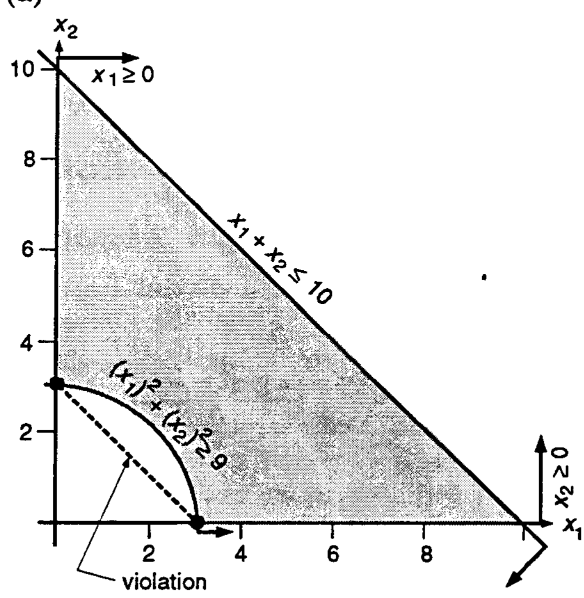


3-14. **(a)** Unimodal because the direction (right or left) from every x to any other with greater objective value is an improving direction. **(b)** Not unimodal because the (right) direction from local minimum $x^{(1)} = 0.5$ towards global minimum $x^{(2)} = 4$ does not improve at $x^{(1)}$. **(c)** Unimodal because the direction $\Delta\mathbf{x}$ from every $\mathbf{x}$ to any other with greater objective value is an improving direction. **(d)** Not unimodal because the direction from $(0,5)$ to better point $(1,3)$ does not improve at $(0,5)$. **(e)** Not unimodal because the direction from $(1,3)$ to better point $(2,1)$ does not improve at $(1,3)$. **(f)** Unimodal because linear. **(g)** Unimodal because linear.

3-15. **(a)** $(3,1,0) + \lambda(-3,3,9)$, $\lambda \in [0,1]$. Setting $\lambda = 1/3$ in this expression yields $\mathbf{z}^{(3)}$; no λ gives $\mathbf{z}^{(4)}$. **(b)** $(2,1,4) + \lambda(5,-1,-5)$, $\lambda \in [0,1]$. Setting $\lambda = 4/5$ in this expression yields $\mathbf{z}^{(3)}$; no λ gives $\mathbf{z}^{(4)}$.

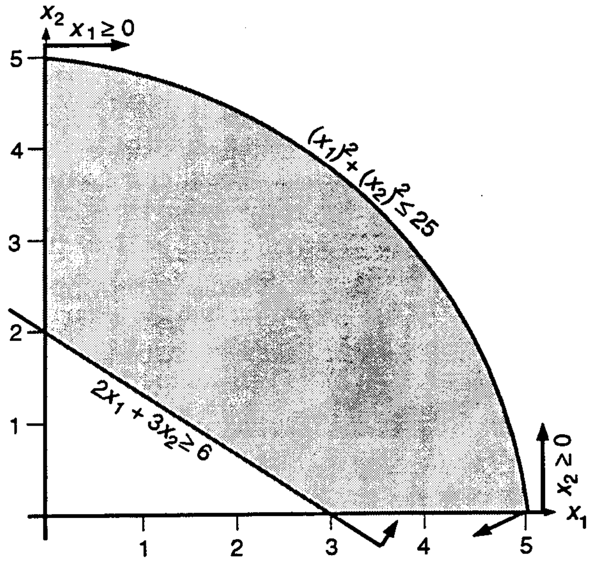
3-16.

(a)



From the graph, the set is not convex because part of the line segment from $\mathbf{x}^{(1)} = (0,3)$ to $\mathbf{x}^{(2)} = (3,0)$ lies outside the feasible region.

(b)



From the graph, the set is convex because the line segment between every pair of feasible solutions lies entirely within the feasible region.
(c) Convex because all constraints are linear.
(d) Convex because all constraints are linear.
(e) Not convex because fractional solutions between say $\mathbf{x}^{(1)} = (0,0,0,4)$ and $\mathbf{x}^{(2)} = (0,0,0,5)$ are infeasible.
(f) Not convex because fractional solutions between say $\mathbf{x}^{(1)} = (1,1,0,0)$ and $\mathbf{x}^{(2)} = (1,1,1,0)$ are infeasible.

3-17. **(a)** Only the first and third constraints are violated at $\mathbf{w} = \mathbf{0}$. Adding nonnegative artificial variables w_4 in the first and w_5 in the third, and minimizing their sum, produces Phase I model: min $w_4 + w_5$, s.t. $40w_1 + 30w_2 + 10w_3 + w_4 = 150$, $w_1 - w_2 \leq 0$, $4w_2 + w_3 + w_5 \geq 10$, $w_1, w_2, w_3, w_4, w_5 \geq 0$. Setting $w_4 = 150 - 40(0) - 30(0) - 10(0) = 150$ and $w_5 = 10 - 4(0) - (0) = 10$ (or any higher value) completes a starting (artificially) feasible solution.
(b) Only the second and third constraints are violated at $\mathbf{w} = \mathbf{0}$. Adding nonnegative artificial variable w_4 in the second, subtracting w_5 in the third (negative RHS), and minimizing their sum, produces Phase I model: min $w_4 + w_5$, s.t. $3w_1 + w_2 + 2w_3 \leq 9$, $4w_1 + 4w_2 + w_4 \geq 6$, $w_1 - w_2 - w_3 - w_5 = -2$, $w_1, w_2, w_3, w_4, w_5 \geq 0$. Setting $w_4 = 6 - 4(0) - 4(0) = 6$ (or any higher value) and $w_5 = -(2 - (0) + (0) + (0)) = 2$ completes a starting (artificially) feasible solution.
(c) All three constraints are violated at $\mathbf{w} = \mathbf{0}$. Subtracting nonnegative artificial variable w_4 in the first, adding w_5 in the second, adding w_6 in the third, and minimizing their sum, produces Phase I model: min $w_3 + w_4 + w_5$, s.t. $(w_1-3)^2+(w_2-3)^2-w_3 \leq 4$, $2w_1+2w_2+w_4 = 5$, $w_1 + w_5 \geq 3$, $w_3, w_4, w_5 \geq 0$. Setting $w_3 = (3-0)^2+(3-0)^2-4 = 14$ (or any larger value), $w_4 = 5 - 2(0) = 2(0) = 5$, and $w_5 = 3 - (0) = 3$ (or any larger value) completes a starting (artificially) feasible solution.
(d) Only the third constraint is violated at $\mathbf{w} = \mathbf{0}$. Adding nonnegative artificial variable w_3 there and minimizing its value produces Phase I model: min w_3, s.t. $(w_1 - 5)^2 + (w_2)^2 \leq 25$, $3w_1 - w_2 = 0$, $w_1 + w_3 \geq 2$, $w_2, w_3 \geq 0$. Setting $w_3 = 2-(0) = 2$ (or any greater value) completes a starting (artificially) feasible solution.

3-18. **(a)** Stop and conclude the model is infeasible because all artificial variables cannot be eliminated. **(b)** Drop artificial variables and proceed with Phase II from initial solution $\mathbf{y} = (22,4,3)$ which is feasible because both artificials $= 0$ at the end of Phase I. **(c)** Drop artificial variables and proceed with Phase II from initial solution $\mathbf{y} = (1,3,1)$ which is feasible because both artificials $= 0$ at the end of Phase I. **(d)** With some artificials positive, the current (y_1, y_2, y_3) solution is not feasible. But there may still be such a solution because the optimum is only local. Repeat Phase I from a new starting point.

3-19. Needed artificial variables and their starting (artificially) feasible values are exactly as in Exercise 3-17. **(a)** For a maximize model, subtract a large multiple of the artificial variables in the objective to obtain big-M model: max $22w_1-w_2+15w_3-M(w_4+w_5)$, s.t. $40w_1+30w_2+10w_3+w_4 = 150$, $w_1-w_2 \leq 0$, $4w_2+w_3+w_5 \geq 10$, $w_1, w_2, w_3, w_4, w_5 \geq 0$; $w_4 = 150$, $w_5 = 10$
(b) For a minimize model, add a large multiple of the artificial variables in the objective to obtain big-M model: min $-11w_1+10w_2+w_3+M(w_4+w_5)$, s.t. $3w_1+w_2+2w_3 \leq 9$, $4w_1+4w_2+w_4 \geq 6$, $w_1 - w_2 - w_3 - w_5 = -2$, $w_1, w_2, w_3, w_4, w_5 \geq 0$;

$w_4 = 6$, $w_5 = 2$
(c) For a minimize model, add a large multiple of the artificial variables in the objective to obtain big-M model: min $2w_1+3w_2+M(w_3+w_4+w_5)$, s.t. $(w_1-3)^2+(w_2-3)^2-w_3 \le 4$, $2w_1+2w_2+w_4 = 5$, $w_1+w_5 \ge 3$, $w_3, w_4, w_5 \ge 0$; $w_3 = 14$, $w_4 = 5$, $w_5 = 3$
(d) For a maximize model, subtract a large multiple of the artificial variables in the objective to obtain big-M model: max $w_1w_2 - Mw_3$, s.t. $(w_1-5)^2+(w_2)^2 \le 25$, $3w_1-w_2 = 0$, $w_1+w_3 \ge 2$, $w_2, w_3 \ge 0$; $w_3 = 2$

3-20. **(a)** Stop and conclude the model is infeasible if M is big enough because artificial variables remain positive. Otherwise, increase M and repeat the search. **(b)** Stop and conclude $\mathbf{y} = (22, 4, 3)$ is a global optimum for the original model because it is feasible with all artificials =0 and optimal because a global optimum was obtained with the big-M model. **(c)** Conclude $\mathbf{y} = (1, 3, 1)$ is a local optimum for the original model because it is feasible with all artificials =0 but possibly not a global optimum because the big-M search yielded only a local. If desired, repeat the big-M search from a new starting point. **(d)** Conclude nothing because only a local optimum has been obtained and some artificials remain positive. Repeat the big-M search from a new starting point.