A Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

Second Exam, Friday, November 7, 2002.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts one hundred minutes.

Q1		/ 20
Q2		/ 15
Q3		/ 25
Q4		/ 20
Q5		/ 20
Total		/ 100
Grade		

SOLUTIONS

1. (20 points) Up to 10000 seats are to be divided between the media, the competing universities and the general public. At least half as many tickets should go to the competing universities as to the general public. The problem can be modeled using the linear program

$$\begin{array}{llcrcrcl} \max & & & 45x_2 & + & 100x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 10000 \\ & & & x_2 & - & \frac{1}{2}x_3 & \geq & 0 \\ & x_1 & & & & & \geq & 500 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

where x_1, x_2, and x_3 denote the number of seats assigned to the media, to the competing universities, and to the general public, respectively.

(a) (10 points) Find the dual linear program to the linear program

(b) (10 points) To accommodate high demand from student supporters of the participating universities, the NCAA is considering marketing a new "scrunch" seat that consumes only 80% of a regular bleacher seat but counts fully against the "university $\geq$ half public" rule. Could an optimal solution allocate any such seats at a ticket price of \$35? At a price of \$25? (Hint: If you formulate the dual problem as in the text, the optimal dual solution is $y_1 = 81.667$, $y_2 = -36.667$, and $y_3 = -81.667$.)

(a)

$$\begin{array}{llcrcl} \min & 10000y_1 & & +500y_3 & & \\ \text{s.t.} & y_1 & & +y_3 & \geq & 0 \\ & y_1 & +y_2 & & \geq & 45 \\ & y_1 & -\frac{1}{2}y_2 & & \geq & 100 \\ & y_1 \geq 0, & y_2, y_3 \leq 0 & & & \end{array}$$

(b) Add new primal column, cost α.

Get extra dual constraint:

$$0.8y_1 + y_2 \geq \alpha$$

We have $0.8y_1 + y_2 = \frac{4}{5}(81\tfrac{2}{3}) - 36\tfrac{2}{3} = 28\tfrac{2}{3}$.

So dual feasible if $\alpha = 25$, that is, optimal to keep old solution.

Not dual feasible if $\alpha = 35$, that is, want to sell some seats at this price.

2. (15 points; no partial credit, each part is worth five points) A resource allocation problem of the form

$$\begin{array}{ll} \max & c^T x \\ \text{st} & Ax \le b \\ & x \ge 0 \end{array}$$

has optimal tableau

$$M^* = \begin{array}{c|ccccccc} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 \\ \hline 20 & 0 & 0 & 2 & 0 & 3 & 1 & 4 \\ \hline 30 & 0 & 0 & 1 & 1 & 0 & 2 & -3 \\ 40 & 1 & 0 & 2 & 0 & 1 & -4 & -2 \\ 20 & 0 & 1 & -1 & 0 & -1 & 5 & 4 \end{array}$$

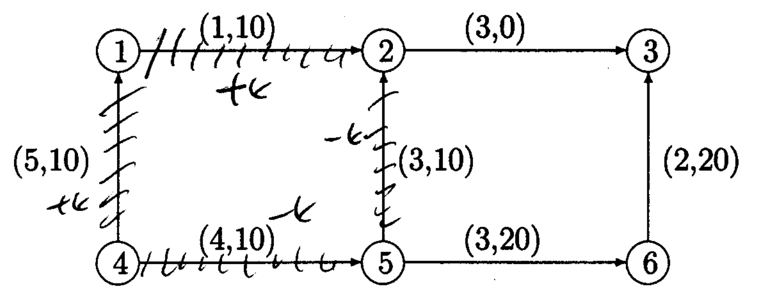
The problem has four products and three resources, so s_i represents the amount of resource i which is unused and x_i represents the amount of product i which is produced.

(a) Suppose somebody offered to buy a small amount of resource 1. How much would you charge for each unit of the resource? Circle your choice:
1 **2** **3** **4**
None of the above

(b) What is the upper bound k on the number of units of the resource you would sell at this price? Circle your choice:
4 **5** **10** **15** **20** **30** **40**
None of the above

(c) Now suppose that you want to sell slightly more than k units of this resource. Would you charge more or less than your answer in part (a) for additional units? Circle your choice:
Charge more than in part (a) **Charge less than in part (a)**
Charge the same as in part (a)

3. (25 points) Consider the network flow problem shown below, with the indicated feasible solution. The numbers on each arc give the cost c_{ij} and the flow x_{ij} as (c_{ij}, x_{ij}).



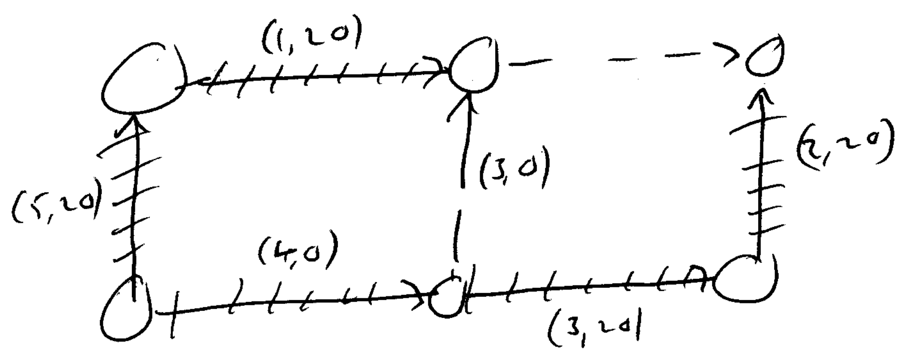
Node 1:	transshipment	Node 4:	supply 20
Node 2:	demand 20	Node 5:	supply 20
Node 3:	demand 20	Node 6:	transshipment

(a) (15 points) Find a basic feasible solution that is at least as good as the given solution.

(b) (10 points) Is the basic feasible solution you found optimal?

(a) Modify flow around the cycle, by $t = 10$.

Get basic feas soln:



(b) By complementary slackness, $c_{ij} = y_i - y_j$ for all basic arcs.

Set $y_3 = 0$. Then $y_6 = 2 \Rightarrow y_5 = 5 \Rightarrow y_4 = 9$

$\Rightarrow y_1 = 4 \Rightarrow y_2 = 3$

Check nonbasic arcs: $c_{ij} \overset{?}{\geq} y_i - y_j$

$(5,2)$: $c_{52} - y_5 + y_2 = 3 - 5 + 3 = 1$ ✓

$(2,3)$: $c_{23} - y_2 + y_3 = 3 - 3 + 0 = 0$ ✓

So, Yes OPTIMAL.

4. (20 points; no partial credit, each part is worth four points) Consider the primal-dual pair of linear programming problems

$$\begin{array}{llllllll} \min & 9x_1 & + & cx_2 & + & 3x_3 & & & (P)\\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & = & b_1 \\ & 2x_1 & + & x_2 & - & x_3 & = & b_2 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}
\qquad
\begin{array}{llllll} \max & b_1y_1 & + & b_2y_2 & & \\ \text{s.t.} & y_1 & + & 2y_2 & \leq & 9 \quad (D) \\ & 2y_1 & + & y_2 & \leq & c \\ & y_1 & - & y_2 & \leq & 3 \end{array}$$

where c, b_1, and b_2 are constants. Assume we have a nondegenerate basic feasible solution with x_1 and x_3 basic.

(a) For complementary slackness to hold, must we have $y_1 + 2y_2 = 9$? **YES** **NO**

(b) For complementary slackness to hold, must we have $2y_1 + y_2 = c$? **YES** **NO**

(c) For complementary slackness to hold, must we have $y_1 - y_2 = 3$? **YES** **NO**

(d) Which of the following solutions satisfies complementary slackness? Circle exactly one:
A: $y_1 = 3,\ y_2 = 3$ **B:** $y_1 = \frac{c}{3} + 1,\ y_2 = \frac{c}{3} - 2$
C: $y_1 = 5,\ y_2 = 2$ **D:** $y_1 = c - 1,\ y_2 = -c + 2$
E: None of the above

(e) For what range of c is this solution optimal? Circle exactly one:
A: $c \leq 7$ **B:** $c \geq 6$ **C:** $6 \leq c \leq 12$ **D:** $c \geq 12$
E: $c \geq 9$ **F: For any** c **G: For no** c **H:** $c = 8$
I: None of the above

$2y_1 + y_2 = 12$. So need $c \geq 12$ for dual feasibility (and primal optimality)

5. (20 points, no partial credit, each part worth four points) For each part, either the statement is always true or there exists a counterexample to the statement. If the statement is always true, circle **TRUE**, otherwise circle **FALSE**.

   (a) A multicommodity network flow problem where the demands are integer and the arc capacities are integer will always have an integer optimal solution.

   TRUE **FALSE**

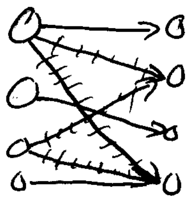
   (b) The point $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 2$, $x_5 = 0$ is optimal for the linear program

$$\begin{array}{lrcrcrcrcl} \min & & & 5x_3 & + & 2x_4 & + & 3x_5 & & \\ \text{subject to} & x_1 & & - \; x_3 & - & x_4 & - & x_5 & = & -2 \\ & & x_2 & + \; 2x_3 & + & 3x_4 & - & 2x_5 & = & 7 \\ & & & & & & & x_i & \geq & 0 \quad \text{for } i = 1, \ldots, 5. \end{array}$$

Dual simplex

   TRUE **FALSE**

   (c) Consider the transportation problem represented in the following table:



	20	30	30	40
40	3^{20}	2^{10}	9	9^{10}
30	9	4	1^{30}	2
30	7	9^{20}	9	4^{10}
20	9	3	5	8^{20}

   The given solution is a basic feasible solution.

   TRUE **FALSE**

   (d) Consider the transportation problem represented in the following table:

	20	30	30	40
40	3^{20}	2^{20}	9	9
30	9	4^{10}	1^{20}	2
30	7	9	9^{10}	4^{20}
20	9	3	5	8^{20}

   The Northwest Corner Rule will give an initial basic feasible solution with $x(3,3) = 10$.

   TRUE **FALSE**

   (e) Consider the transportation problem represented in the following table:

$u_i + v_j = c_{ij}$

		$v_1 = 3$	2	$\cancel{4}$ -1	0
		20	30	30	40
$u_1 = 0$	40	3^{10}	2^{30}	$\cancel{9}_{10}$	$\cancel{9}_{9}$
2	30	$\cancel{9}_{4}$	$\cancel{4}_{0}$	1^{10}	2^{20}
4	30	7^{10}	$\cancel{9}_{3}$	$\cancel{9}_{6}$	4^{20}
6	20	$\cancel{9}_{0}$	$\cancel{3}_{-5}$	5^{20}	$\cancel{8}_{2}$

   The given basic feasible solution is optimal.

False: introduce (4,2).

   TRUE **FALSE**