12. $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{bmatrix}$ **13.** $A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & -2 & 1 \\ 3 & 1 & -1 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ **15.** $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 2 \\ 2 & 1 & 4 \end{bmatrix}$

16. $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 2 \end{bmatrix}$ **17.** $A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 3 & 0 & 0 \\ 0 & 4 & 1 & 2 \\ 0 & 3 & 1 & 4 \end{bmatrix}$

18. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

19. $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 4 & 1 & 4 \end{bmatrix}$

20. Let $A = (a_{ij})$ be a given (3×3) matrix. Form the associated (3×5) matrix B shown next:

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

a) Subtract the sum of the three upward diagonal products from the sum of the three downward diagonal products and argue that your result is equal to $\det(A)$.

b) Show, by example, that a similar basketweave algorithm cannot be used to calculate the determinant of a (4×4) matrix.

In Exercises 21 and 22, find all ordered pairs (x, y) such that A is singular.

21. $A = \begin{bmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ **22.** $A = \begin{bmatrix} x & 1 & 1 \\ 2 & 1 & 1 \\ 0 & -1 & y \end{bmatrix}$

23. Let $A = (a_{ij})$ be the $(n \times n)$ matrix specified thus: $a_{ij} = d$ for $i = j$ and $a_{ij} = 1$ for $i \neq j$. For $n = 2$, 3, and 4, show that

$$\det(A) = (d-1)^{n-1}(d-1+n).$$

24. Let A and B be $(n \times n)$ matrices. Use Theorems 2 and 3 to give a quick proof of each of the following.

a) If either A or B is singular, then AB is singular.

b) If AB is singular, then either A or B is singular.

25. Suppose that A is an $(n \times n)$ nonsingular matrix, and recall that $\det(I) = 1$, where I is the $(n \times n)$ identity matrix. Show that $\det(A^{-1}) = 1/\det(A)$.

26. If A and B are $(n \times n)$ matrices, then usually $AB \neq BA$. Nonetheless, argue that always $\det(AB) = \det(BA)$.

In Exercises 27–30, use Theorem 2 and Exercise 25 to evaluate the given determinant, where A and B are $(n \times n)$ matrices with $\det(A) = 3$ and $\det(B) = 5$.

27. $\det(ABA^{-1})$ **28.** $\det(A^2B)$

29. $\det(A^{-1}B^{-1}A^2)$ **30.** $\det(AB^{-1}A^{-1}B)$

31. a) Let A be an $(n \times n)$ matrix. If $n = 3$, $\det(A)$ can be found by evaluating three (2×2) determinants. If $n = 4$, $\det(A)$ can be found by evaluating twelve (2×2) determinants. Give a formula, $H(n)$, for the number of (2×2) determinants necessary to find $\det(A)$ for an arbitrary n.

b) Suppose you can perform additions, subtractions, multiplications, and divisions each at a rate of one per second. How long does it take to evaluate $H(n)$ (2×2) determinants when $n = 2$, $n = 5$, and $n = 10$?

32. Let U and V be $(n \times n)$ upper-triangular matrices. Prove a special case of Theorem 2: $\det(UV) = \det(U)\det(V)$. [*Hint:* Use the definition for matrix multiplication to calculate the diagonal entries of the product UV, and then apply Theorem 4. You will also need to recall from Exercise 67 in Section 1.5 that UV is an upper-triangular matrix.]

33. Let V be an $(n \times n)$ triangular matrix. Use Theorem 4 to prove that $\det(V^T) = \det(V)$.

34. Let $T = (t_{ij})$ be an $(n \times n)$ upper-triangular matrix. Prove that $\det(T) = t_{11}t_{22}\ldots t_{nn}$. [*Hint:* Use mathematical induction, beginning with a (2×2) upper-triangular determinant.]

9. $\begin{vmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 4 & 1 & 3 \\ 2 & 1 & 5 & 6 \end{vmatrix}$ **10.** $\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 3 & 1 & 2 \end{vmatrix}$

11. $\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 6 & 3 \\ 2 & 4 & 1 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ **12.** $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 2 & 1 & 0 & 6 \\ 3 & 2 & 2 & 4 \end{vmatrix}$

In Exercises 13–18, assume that the (3×3) matrix A satisfies $\det(A) = 2$, where A is given by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Calculate $\det(B)$ in each case.

13. $B = \begin{bmatrix} a & b & 3c \\ d & e & 3f \\ g & h & 3i \end{bmatrix}$ **14.** $B = \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$

15. $B = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}$

16. $B = \begin{bmatrix} a & b & c \\ a+d & b+e & c+f \\ g & h & i \end{bmatrix}$

17. $B = \begin{bmatrix} d & e & f \\ 2a & 2b & 2c \\ g & h & i \end{bmatrix}$

18. $B = \begin{bmatrix} d & f & e \\ a & c & b \\ g & i & h \end{bmatrix}$

In Exercises 19–22, evaluate the (4×4) determinants. Theorems 6–8 can be used to simplify the calculations.

19. $\begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 1 \end{vmatrix}$ **20.** $\begin{vmatrix} 0 & 2 & 1 & 3 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 2 & 2 & 1 & 2 \end{vmatrix}$

21. $\begin{vmatrix} 0 & 4 & 1 & 3 \\ 0 & 2 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 4 \end{vmatrix}$ **22.** $\begin{vmatrix} 2 & 2 & 4 & 4 \\ 1 & 1 & 3 & 3 \\ 1 & 0 & 2 & 1 \\ 4 & 1 & 3 & 2 \end{vmatrix}$

In Exercises 23 and 24, use row operations to obtain a triangular determinant and find the value of the original Vandermonde determinant.

23. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ **24.** $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$

25. Let A be an $(n \times n)$ matrix. Use Theorem 7 to argue that $\det(cA) = c^n \det(A)$.

26. Prove the corollary to Theorem 6. [*Hint:* Suppose that the ith and jth rows of A are identical. Interchange these two rows and let B denote the matrix that results. How are $\det(A)$ and $\det(B)$ related?]

27. Find examples of (2×2) matrices A and B such that $\det(A + B) \neq \det(A) + \det(B)$.

28. An $(n \times n)$ matrix A is called ***skew symmetric*** if $A^T = -A$. Show that if A is skew symmetric, then $\det(A) = (-1)^n \det(A)$. [*Hint:* Use Theorem 5 and Exercise 25.] Now, argue that an $(n \times n)$ skew-symmetric matrix is singular when n is an odd integer.

ultimately requires the evaluation of $n!/2$ determinants of order (2×2). Even for modest values of n, the number $n!/2$ is alarmingly large. For instance,

$$10!/2 = 1{,}814{,}400,$$

whereas

$$20!/2 > 1.2 \times 10^{18}.$$

The enormous number of calculations required to compute $\det(A - tI)$ means that we cannot find $p(t)$ in any practical sense by expanding $\det(A - tI)$. In Chapter 6, we note that there are relatively efficient ways of finding $\det(A)$, but these techniques (which amount to using elementary row operations to triangularize A) are not useful in our problem of computing $\det(A - tI)$ because of the variable t. In Section 7.3, we resolve this difficulty by using similarity transformations to transform A to a matrix H, where A and H have the same characteristic polynomial, and where it is a trivial matter to calculate the characteristic polynomial for H. Moreover, these transformation methods will give us some other important results as a by-product, results such as the Cayley–Hamilton theorem, which have some practical computational significance.

4.4 EXERCISES

In Exercises 1–14, find the characteristic polynomial and the eigenvalues for the given matrix. Also, give the algebraic multiplicity of each eigenvalue. [*Note:* In each case the eigenvalues are integers.]

1. $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 13 & -16 \\ 9 & -11 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -6 & -1 & 2 \\ 3 & 2 & 0 \\ -14 & -2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$

10. $\begin{bmatrix} -7 & 4 & -3 \\ 8 & -3 & 3 \\ 32 & -16 & 13 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix}$

13. $\begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

15. Prove property (b) of Theorem 11. [*Hint:* Begin with $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \boldsymbol{\theta}$.]

16. Prove property (c) of Theorem 11.

17. Complete the proof of property (a) of Theorem 11.

18. Let $q(t) = t^3 - 2t^2 - t + 2$; and for any $(n \times n)$ matrix H, define the matrix polynomial $q(H)$ by

$$q(H) = H^3 - 2H^2 - H + 2I,$$

where I is the $(n \times n)$ identity matrix.

a) Prove that if λ is an eigenvalue of H, then the number $q(\lambda)$ is an eigenvalue of the matrix $q(H)$. [*Hint:* Suppose that $H\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x} \neq \boldsymbol{\theta}$, and use Theorem 11 to evaluate $q(H)\mathbf{x}$.]

b) Use part a) to calculate the eigenvalues of $q(A)$ and $q(B)$, where A and B are from Exercises 7 and 8, respectively.

19. With $q(t)$ as in Exercise 18, verify that $q(C)$ is the zero matrix, where C is from Exercise 9. (Note that $q(t)$ is the characteristic polynomial for C. See Exercises 20–23.)

Exercises 20–23 illustrate the ***Cayley–Hamilton theorem***, which states that if $p(t)$ is the characteristic polynomial for A, then $p(A)$ is the zero matrix. (As in Exercise 18, $p(A)$ is the $(n \times n)$ matrix that comes from substituting A for t in $p(t)$.) In Exercises 20–23, verify that $p(A) = \mathcal{O}$ for the given matrix A.

20. A in Exercise 3

21. A in Exercise 4

22. A in Exercise 9

23. A in Exercise 13

24. This problem establishes a special case of the Cayley–Hamilton theorem.

- **a)** Prove that if B is a (3×3) matrix, and if $B\mathbf{x} = \boldsymbol{\theta}$ for every $\mathbf{x}$ in R^3, then B is the zero matrix. [*Hint:* Consider $B\mathbf{e}_1$, $B\mathbf{e}_2$, and $B\mathbf{e}_3$.]
- **b)** Suppose that λ_1, λ_2, and λ_3 are the eigenvalues of a (3×3) matrix A, and suppose that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are corresponding eigenvectors. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set, and if $p(t)$ is the characteristic polynomial for A, then $p(A)$ is the zero matrix. [*Hint:* Any vector $\mathbf{x}$ in R^3 can be expressed as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.]

25. Consider the (2×2) matrix A given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The characteristic polynomial for A is $p(t) = t^2 - (a+d)t + (ad - bc)$. Verify the Cayley–Hamilton theorem for (2×2) matrices by forming A^2 and showing that $p(A)$ is the zero matrix.

26. Let A be the (3×3) upper-triangular matrix given by

$$A = \begin{bmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{bmatrix}.$$

The characteristic polynomial for A is $p(t) = -(t-a)(t-b)(t-c)$. Verify that $p(A)$ has the form $p(A) = -(A - aI)(A - bI)(A - cI)$. [*Hint:* Expand $p(t)$ and $p(A)$; for instance, $(A - bI)(A - cI) = A^2 - (b+c)A + bcI$]. Next, show that $p(A)$ is the zero matrix by forming the product of the matrices $A - aI$, $A - bI$, and $A - cI$. [*Hint:* Form the product $(A - bI)(A - cI)$ first.]

27. Let $q(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, and define the $(n \times n)$ "companion" matrix by

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

- **a)** For $n = 2$ and for $n = 3$, show that $\det(A - tI) = (-1)^n q(t)$.
- **b)** Give the companion matrix A for the polynomial $q(t) = t^4 + 3t^3 - t^2 + 2t - 2$. Verify that $q(t)$ is the characteristic polynomial for A.
- **c)** Prove for all n that $\det(A - tI) = (-1)^n q(t)$.

28. The power method is a numerical method used to estimate the dominant eigenvalue of a matrix A. (By the dominant eigenvalue, we mean the one that is largest in absolute value.) The algorithm proceeds as follows:

- **a)** Choose any starting vector $\mathbf{x}_0$, $\mathbf{x}_0 \neq \boldsymbol{\theta}$.
- **b)** Let $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $k = 0, 1, 2, \ldots$.
- **c)** Let $\beta_k = \mathbf{x}_k^T \mathbf{x}_{k+1} / \mathbf{x}_k^T \mathbf{x}_k$, $k = 0, 1, 2, \ldots$.

Under suitable conditions, it can be shown that $\{\beta_k\} \to \lambda_1$, where λ_1 is the dominant eigenvalue of A. Use the power method to estimate the dominant eigenvalue of the matrix in Exercise 9. Use the starting vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and calculate β_0, β_1, β_2, β_3, and β_4.

29. This exercise gives a condition under which the power method (see Exercise 28) converges. Suppose that A is an $(n \times n)$ matrix and has real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Furthermore, suppose that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$, and the starting vector $\mathbf{x}_0$ satisfies $\mathbf{x}_0 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$, where $c_1 \neq 0$. Prove that

$$\lim_{k \to \infty} \beta_k = \lambda_1.$$

[*Hint:* Observe that $\mathbf{x}_j = A^j \mathbf{x}_0$, $j = 1, 2, \ldots$, and use Theorem 11 to calculate $\mathbf{x}_{k+1}$ and $\mathbf{x}_k$. Next, factor all powers of λ_1 from the numerator and denominator of $\beta_k = \mathbf{x}_k^T \mathbf{x}_{k+1} / \mathbf{x}_k^T \mathbf{x}_k$.]