4.5 EXERCISES

The following list of matrices and their respective characteristic polynomials is referred to in Exercises 1–11.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix},$$

$$p(t) = (t-3)(t-1), \qquad p(t) = (t-2)^2,$$

$$C = \begin{bmatrix} -6 & -1 & 2 \\ 3 & 2 & 0 \\ -14 & -2 & 5 \end{bmatrix}, \qquad D = \begin{bmatrix} -7 & 4 & -3 \\ 8 & -3 & 3 \\ 32 & -16 & 13 \end{bmatrix},$$

$$p(t) = -(t-1)^2(t+1), \qquad p(t) = -(t-1)^3,$$

$$E = \begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix},$$

$$p(t) = (t+1)(t+5)^2(t-15), \qquad p(t) = (t+2)(t-2)^3$$

In Exercises 1–11, find a basis for the eigenspace E_λ for the given matrix and the value of λ. Determine the algebraic and geometric multiplicities of λ.

1. $A, \lambda = 3$ **2.** $A, \lambda = 1$ **3.** $B, \lambda = 2$

4. $C, \lambda = 1$ **5.** $C, \lambda = -1$ **6.** $D, \lambda = 1$

7. $E, \lambda = -1$ **8.** $E, \lambda = 5$ **9.** $E, \lambda = 15$

10. $F, \lambda = -2$ **11.** $F, \lambda = 2$

In Exercises 12–17, find the eigenvalues and the eigenvectors for the given matrix. Is the matrix defective?

12. $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ **13.** $\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **15.** $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} -1 & 6 & 2 \\ 0 & 5 & -6 \\ 1 & 0 & -2 \end{bmatrix}$ **17.** $\begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$

18. If a vector $\mathbf{x}$ is a linear combination of eigenvectors of a matrix A, then it is easy to calculate the product $\mathbf{y} = A^k\mathbf{x}$ for any positive integer k. For instance, suppose that $A\mathbf{u}_1 = \lambda_1\mathbf{u}_1$ and $A\mathbf{u}_2 = \lambda_2\mathbf{u}_2$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are nonzero vectors. If $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$, then (see Theorem 11 of Section 4.4) $\mathbf{y} = A^k\mathbf{x} = A^k(a_1\mathbf{u}_1 + a_2\mathbf{u}_2) = a_1A^k\mathbf{u}_1 + a_2A^k\mathbf{u}_2 = a_1(\lambda_1)^k\mathbf{u}_1 + a_2(\lambda_2)^k\mathbf{u}_2$. Find $A^{10}\mathbf{x}$, where

$$A = \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}.$$

19. As in Exercise 18, calculate $A^{10}\mathbf{x}$ for

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -2 \\ 0 & 6 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}.$$

20. Consider a (4×4) matrix H of the form

$$H = \begin{bmatrix} \times & \times & \times & \times \\ a & \times & \times & \times \\ 0 & b & \times & \times \\ 0 & 0 & c & \times \end{bmatrix} \qquad (11)$$

In matrix (11) the entries designated $\times$ may be zero or nonzero. Suppose, in matrix (11), that a, b, and c are nonzero. Let λ be any eigenvalue of H. Show that the geometric multiplicity of λ is equal to 1. [*Hint:* Verify that the rank of $H - \lambda I$ is exactly equal to 3.]

21. An $(n \times n)$ matrix P is called ***idempotent*** if $P^2 = P$. Show that if P is an invertible idempotent matrix, then $P = I$.

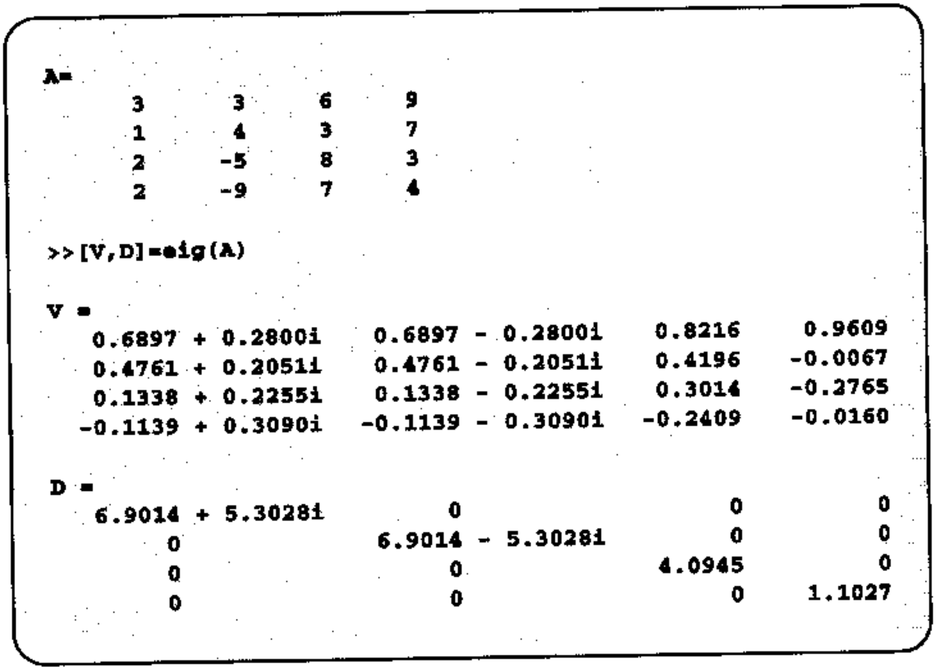
22. Let P be an idempotent matrix. Show that the only eigenvalues of P are $\lambda = 0$ and $\lambda = 1$. [*Hint:* Suppose that $P\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \boldsymbol{\theta}$.]

23. Let $\mathbf{u}$ be a vector in R^n such that $\mathbf{u}^T\mathbf{u} = 1$. Show that the $(n \times n)$ matrix $P = \mathbf{u}\mathbf{u}^T$ is an idempotent matrix. [*Hint:* Use the associative properties of matrix multiplication.]

24. Verify that if Q is idempotent, then so is $I - Q$. Also verify that $(I - 2Q)^{-1} = I - 2Q$.

25. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n such that $\mathbf{u}^T\mathbf{u} = 1$, $\mathbf{v}^T\mathbf{v} = 1$, and $\mathbf{u}^T\mathbf{v} = 0$. Show that $P = \mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T$ is idempotent.

26. Show that any nonzero vector of the form $a\mathbf{u} + b\mathbf{v}$ is an eigenvector corresponding to $\lambda = 1$ for the matrix P in Exercise 25.



```
A=
       3       3       6       9
       1       4       3       7
       2      -5       8       3
       2      -9       7       4

>>[V,D]=eig(A)

V =
   0.6897 + 0.2800i   0.6897 - 0.2800i   0.8216    0.9609
   0.4761 + 0.2051i   0.4761 - 0.2051i   0.4196   -0.0067
   0.1338 + 0.2255i   0.1338 - 0.2255i   0.3014   -0.2765
  -0.1139 + 0.3090i  -0.1139 - 0.3090i  -0.2409   -0.0160

D =
   6.9014 + 5.3028i         0                0         0
        0             6.9014 - 5.3028i       0         0
        0                   0             4.0945       0
        0                   0                0      1.1027
```

Figure 4.3 MATLAB was used to find the eigenvalues and eigenvectors of matrix A in Example 6—that is, $AV = VD$ or $V^{-1}AV = D$, where D is diagonal.

4.6 EXERCISES

In Exercises 1–18, $s = 1 + 2i$, $u = 3 - 2i$, $v = 4 + i$, $w = 2 - i$, and $z = 1 + i$. In each exercise, perform the indicated calculation and express the result in the form $a + ib$.

1. $\bar{u}$

2. $\bar{z}$

3. $u + \bar{v}$

4. $\bar{z} + w$

5. $u + \bar{u}$

6. $s - \bar{s}$

7. $v\bar{v}$

8. $u\bar{v}$

9. $s^2 - w$

10. $z^2 w$

11. $\bar{u}w^2$

12. $s(u^2 + v)$

13. u/v

14. v/u^2

15. s/z

16. $(w + \bar{v})/u$

17. $w + iz$

18. $s - iw$

Find the eigenvalues and the eigenvectors for the matrices in Exercises 19–24. (For the matrix in Exercise 24, one eigenvalue is $\lambda = 1 + 5i$.)

19. $\begin{bmatrix} 6 & 8 \\ -1 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}$

21. $\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$

23. $\begin{bmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$

24. $\begin{bmatrix} 1 & -5 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

In Exercises 25 and 26, solve the linear system.

25. $\begin{aligned} (1+i)x + iy &= 5 + 4i \\ (1-i)x - 4y &= -11 + 5i \end{aligned}$

26. $\begin{aligned} (1-i)x - (3+i)y &= -5 - i \\ (2+i)x + (1+2i)y &= 1 + 6i \end{aligned}$

In Exercises 27–30, calculate $\|\mathbf{x}\|$.

27. $\mathbf{x} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ **28.** $\mathbf{x} = \begin{bmatrix} 3+i \\ 2-i \end{bmatrix}$

29. $\mathbf{x} = \begin{bmatrix} 1-2i \\ i \\ 3+i \end{bmatrix}$ **30.** $\mathbf{x} = \begin{bmatrix} 2i \\ 1-i \\ 3 \end{bmatrix}$

In Exercises 31–34, use linear algebra software to find the eigenvalues and the eigenvectors.

31. $\begin{bmatrix} 2 & 2 & 5 \\ 5 & 3 & 7 \\ 1 & 5 & 3 \end{bmatrix}$ **32.** $\begin{bmatrix} 1 & 2 & 8 \\ 8 & 4 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 5 & -1 & 0 & 8 \\ 3 & 6 & 8 & -3 \\ 1 & 1 & 4 & 2 \\ 9 & 7 & 6 & 9 \end{bmatrix}$ **34.** $\begin{bmatrix} 5 & 5 & 4 & 6 \\ 0 & 8 & 6 & 7 \\ 1 & 2 & 3 & 1 \\ 6 & 3 & 8 & 5 \end{bmatrix}$

35. Establish the five properties of the conjugate operation listed in (2).

36. Let A be an $(m \times n)$ matrix, and let B be an $(n \times p)$ matrix, where the entries of A and B may be complex. Use Exercise 35 and the definition of AB to show that $\overline{AB} = \bar{A}\bar{B}$. (By $\bar{A}$, we mean the matrix whose ijth entry is the conjugate of the ijth entry of A.) If A is a real matrix and $\mathbf{x}$ is an $(n \times 1)$ vector, show that $\overline{A\mathbf{x}} = A\bar{\mathbf{x}}$.

37. Let A be an $(m \times n)$ matrix, where the entries of A may be complex. It is customary to use the symbol A^* to denote the matrix

$$A^* = (\bar{A})^T.$$

Suppose that A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix. Use Exercise 36 and the properties of the transpose operation to give a quick proof that $(AB)^* = B^*A^*$.

38. An $(n \times n)$ matrix A is called ***Hermitian*** if $A^* = A$.

- **a)** Prove that a Hermitian matrix A has only real eigenvalues. [*Hint:* Observing that $\bar{\mathbf{x}}^T\mathbf{x} = \mathbf{x}^*\mathbf{x}$, modify the proof of Theorem 17.]
- **b)** Let $A = (a_{ij})$ be an $(n \times n)$ Hermitian matrix. Show that a_{ii} is real for $1 \le i \le n$.

39. Let $p(t) = a_0 + a_1t + \cdots + a_nt^n$, where the coefficients $a_0, a_1, \ldots, a_n$ are all real.

- **a)** Prove that if r is a complex root of $p(t) = 0$, then $\bar{r}$ is also a root of $p(t) = 0$.
- **b)** If $p(t)$ has degree 3, argue that $p(t)$ must have at least one real root.
- **c)** If A is a (3×3) real matrix, argue that A must have at least one real eigenvalue.

40. An $(n \times n)$ real matrix A is called ***orthogonal*** if $A^TA = I$. Let λ be an eigenvalue of an orthogonal matrix A, where $\lambda = r + is$. Prove that $\lambda\bar{\lambda} = r^2 + s^2 = 1$. [*Hint:* First show that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for any vector $\mathbf{x}$.]

41. A real symmetric $(n \times n)$ matrix A is called ***positive definite*** if $\mathbf{x}^TA\mathbf{x} > 0$ for all $\mathbf{x}$ in R^n, $\mathbf{x} \neq \boldsymbol{\theta}$. Prove that the eigenvalues of a real symmetric positive-definite matrix A are all positive.

42. An $(n \times n)$ matrix A is called ***unitary*** if $A^*A = I$. (If A is a real unitary matrix, then A is orthogonal; see Exercise 40.) Show that if A is unitary and λ is an eigenvalue for A, then $|\lambda| = 1$.

SIMILARITY TRANSFORMATIONS AND DIAGONALIZATION

In Chapter 1, we saw that two linear systems of equations have the same solution if their augmented matrices are row equivalent. In this chapter, we are interested in identifying classes of matrices that have the same eigenvalues.

As we know, the eigenvalues of an $(n \times n)$ matrix A are the zeros of its characteristic polynomial,

$$p(t) = \det(A - tI).$$