

A

Name:

Code:

Multivariable Calculus and Matrix Algebra, MATH 2010

Final Exam, 11.30am-2.30pm, Wednesday, May 8, 2002.

You may use four sheets of handwritten notes, but no other sources. Answer the **first three problems** and any **three of the remaining six** problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No books or calculators allowed.

Please enter a code above. This code will be used to display grades. The grades will be available via SIS and from the course web page in the next few days.

Please ring your section below.

8am	9am	1pm	3pm
Emilio	Emilio	Kris	Kris
Castronovo	Castronovo	Farwell	Farwell

SOLUTIONS.

Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
Q9	
Total	
Grade	

Answer completely questions 1, 2, and 3.

1. (15 points) Consider the system of equations $Ax = b$, where

$$A = \begin{bmatrix} 1 & 0 & a & 3 \\ 0 & 1 & 2 & c \\ 3 & 1 & 3a+2 & a+c \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}, \quad \text{and } g = \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

- (a) For what value(s) of a and c does $Ax = b$ have infinitely many solutions?
 (b) For what value(s) of a and c is the system $Ax = b$ inconsistent?
 (c) Let $a = c = 1$. For what value(s) of p , q , and r does the system $Ax = g$ have a unique solution?

(a) Row reduce $[A \mid b \mid g]$ to prepare for parts (a), (b), (c).

$$\begin{bmatrix} 1 & 0 & a & 3 & 1 & p \\ 0 & 1 & 2 & c & 1 & q \\ 3 & 1 & 3a+2 & a+c & 7 & r \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 0 & a & 3 & 1 & p \\ 0 & 1 & 2 & c & 1 & q \\ 0 & 1 & 2 & a+c-9 & 4 & r-3p \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c|c} 1 & 0 & a & 3 & 1 & p \\ 0 & 1 & 2 & c & 1 & q \\ 0 & 0 & 0 & a-9 & 3 & r-3p-q \end{array} \right]$$

So $Ax = b$ has infinitely many solutions provided $a \neq 9$

(b) $Ax = b$ is inconsistent if $a = 9$

(c) If $a = c = 1$: $Ax = g$ never has a unique solution:
 more unknowns than equations.

2. (20 points) Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad \text{and } b = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

- (a) (5 points) What are the eigenvalues of A ?
 (b) (10 points) Find a matrix S that diagonalizes A .
 (c) (5 points) Calculate $A^7 b$. What is $\lim_{k \rightarrow \infty} A^k b$?

(a) Solve $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(-1)^{2+2} \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \quad \text{Cofactor expansion about Row 2}$$

$$= (1-\lambda)((2-\lambda)(2-\lambda) - 1) = (1-\lambda)(\lambda^2 - 4\lambda + 3)$$

$$= (1-\lambda)(\lambda-3)(\lambda-1)$$

So eigenvalues are $\lambda=3$ with algebraic multiplicity 1
 $\lambda=1$ with algebraic multiplicity 2

(b) $\lambda=3$: $A-3I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ Solve $(A-3I)x = 0$: $x = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda=1$: $A-I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Solve $(A-I)x = 0$: $x = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

So $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ diagonalizes A . Check: $AS = S \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$

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$$(c) A^7 b: \quad S^{-1}AS = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with } S \text{ as in part (b)}$$

$$\begin{aligned} \text{So } A^7 b &= S (S^{-1}AS)^7 S^{-1}b \\ &= S \begin{bmatrix} 3^7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1}b \end{aligned}$$

$$\text{Find } S^{-1}b: \quad \text{Solve } Sy = b$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{so } y = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{Thus, } A^7 b = S \begin{bmatrix} 3^7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = S \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = b$$

~~It~~ Note that $Ab = b$, so $A^k b = b$ for all k . So $\lim_{k \rightarrow \infty} A^k b = b$.

3. (20 points)

(a) (10 points) Let

$$u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, v = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}, w = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

For what value(s) of a , b , and c are the three vectors u , v , and w orthogonal?Check u, v orthogonal: $u^T v = 7 - 3 - 4 = 0 \checkmark$ Need $u^T w = 0$, that is, $a + 3b - 2c = 0$ ①Need $v^T w = 0$, that is, $7a - b + 2c = 0$ ②

Solve ① & ②:

$$\begin{bmatrix} 1 & 3 & -2 & 0 \\ 7 & -1 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 - 7R_1} \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -22 & 16 & 0 \end{bmatrix}$$

$$\text{So: } b = \frac{16}{22} c, \quad a = 2c - 3b = \frac{-4}{22} c,$$

$$\text{or } b = \frac{8}{11} c, \quad a = \frac{-2}{11} c$$

$$\text{So: } w = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = s \begin{bmatrix} -2 \\ 8 \\ 11 \end{bmatrix}$$

$$\text{Check: } u^T w = s(-2 + 24 - 22) = 0 \checkmark$$

$$v^T w = s(-14 - 8 + 22) = 0 \checkmark$$

(b) (10 points) Let

$$p = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2 \end{bmatrix}, q = \begin{bmatrix} 7 \\ -1 \\ 2 \\ -9 \end{bmatrix}$$

be a basis for a subspace W of \mathbb{R}^4 . Find an orthogonal basis for W .Replace q by

$$\bar{q} = q - \frac{p^T q}{p^T p} p = \begin{bmatrix} 7 \\ -1 \\ 2 \\ -9 \end{bmatrix} - \frac{-18}{18} \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -1 \\ 2 \\ -9 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 0 \\ -7 \end{bmatrix}$$

$$\text{Check: } p^T \bar{q} = 0 \quad \checkmark$$

Answer any three of the following problems 4 through 9

4. (15 points)

(a) The temperature at the point (x, y) on a metal plate is

$$T = \frac{2x}{x^2 + y^2}$$

- i. (5 points) Find the direction of greatest increase in temperature from the point $(4, 3)$.
- ii. (5 points) Find the directions of no change in temperature from the point $(4, 3)$.

(i) Need to find the gradient of T .

$$\frac{\partial T}{\partial x} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} = \frac{2}{25} - \frac{64}{25^2} = \frac{-14}{625} \text{ at } (4, 3)$$

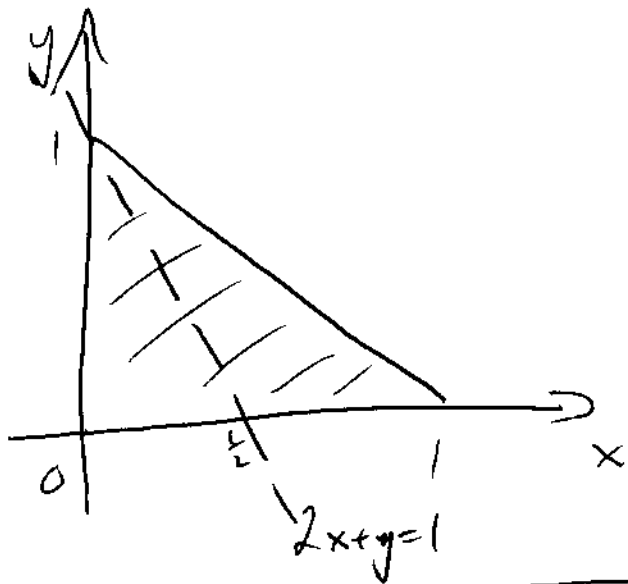
$$\frac{\partial T}{\partial y} = \frac{-4xy}{(x^2 + y^2)^2} = \frac{-48}{625} \text{ at } (4, 3)$$

So direction of greatest increase is $\underline{\underline{\begin{bmatrix} -14/625 \\ -48/625 \end{bmatrix}}}$.

(ii) Directions of no change are orthogonal to the gradient:

$$\underline{\underline{\begin{bmatrix} 48 \\ -14 \end{bmatrix}}} \text{ and } \underline{\underline{\begin{bmatrix} -48 \\ 14 \end{bmatrix}}}$$

- (b) (5 points) Find the absolute extrema of the function $f(x, y) = (2x + y - 1)^2$ over the triangular region in the xy -plane with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.



$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 4(2x+y-1) \\ 2(2x+y-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ when } 2x+y=1$$

Note that $f(x, y) \geq 0$ for all points (x, y) .

So minimized at any point on the straight line between $(\frac{1}{2}, 0)$ and $(0, 1)$, with value 0.

For maximum, check boundaries:

If $x=0$: $f(0, y) = (y-1)^2$, maximized at $y=0$, with value 1.

If $y=0$: $f(x, 0) = (2x-1)^2$, maximized at $x=0$ and $x=1$, again with value 0

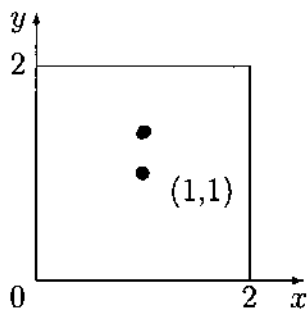
If $y=1-x$: $f(x, 1-x) = (2x + (1-x) - 1)^2 = x^2$,
maximized at $x=1$ (so $y=0$), with value 1

Maximized at $(0, 0)$ and $(1, 0)$ with value 1

5. (15 points) The center of mass of the square lamina with vertices at $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$ with constant density is at the point $(1,1)$.

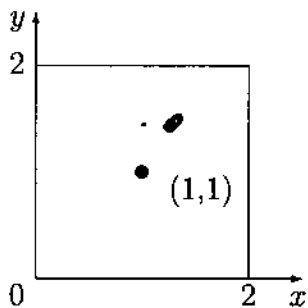
(a) (9 points) Make a conjecture about how the center of mass (\bar{x}, \bar{y}) will change for the nonconstant densities $\rho(x, y)$ given below. Mark your estimate on the picture.

i. $\rho(x, y) = 3y$.



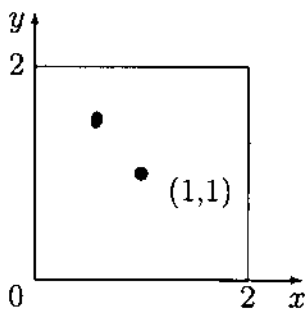
Directly above $(1,1)$

ii. $\rho(x, y) = x^2 + 2y^2$.



To the right and above,
more "above" than "to the right."

iii. $\rho(x, y) = (2 - x)y$.



On the line between $(1,1)$
and $(0,2)$.

(b) (6 points) Find the mass and center of mass for one of the densities in part 2a.

$$D_0 \quad \rho(x, y) = 3y;$$

$$\begin{aligned} \text{Mass} &= \iint_D \rho(x, y) \, dA = \int_0^2 \int_0^2 3y \, dy \, dx \\ &= \int_0^2 \left[\frac{3}{2} y^2 \right]_0^2 \, dx = \int_0^2 6 \, dx = 12 \end{aligned}$$

(= same mass as for a uniform density of 3).

Center of mass: $\bar{x} = 1$ by symmetry

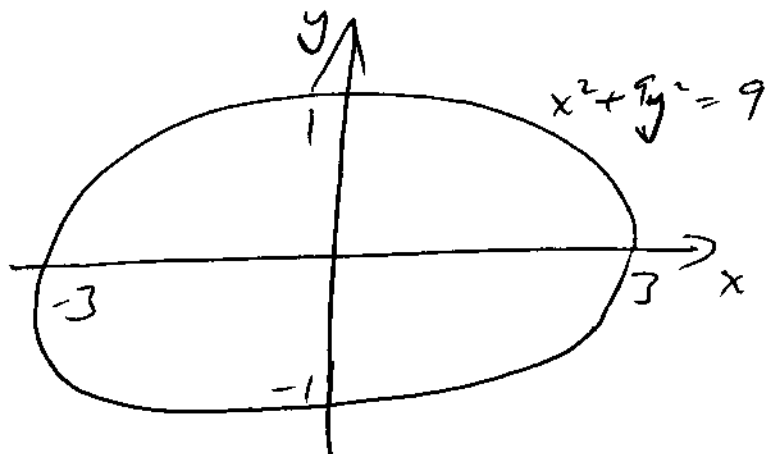
$$\bar{y} = \frac{1}{12} \iint_D y \rho(x, y) \, dA = \frac{1}{12} \int_0^2 \int_0^2 3y^2 \cdot y \, dy \, dx$$

$$= \frac{1}{12} \int_0^2 \left[y^4 \right]_0^2 \, dx = \frac{8}{12} \int_0^2 1 \, dx$$

$$= \frac{4}{3}, \quad \text{as for our conjecture.}$$

$$\text{So, } \boxed{(\bar{x}, \bar{y}) = \left(1, \frac{4}{3}\right)}$$

6. (15 points) Use Green's Theorem to find the area inside the ellipsoid $x^2 + 9y^2 = 9$.



Want $\iint_D 1 \, dA$

Take $Q(x,y) = \frac{1}{2}x$, $P(x,y) = -\frac{1}{2}y$

so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

$$\text{Area} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P \, dx + Q \, dy$$

~~$$\iint_D 1 \, dA = \int_C P \, dx + Q \, dy$$~~

$$= \frac{1}{2} \int_C x \, dy - \frac{1}{2} \int_C y \, dx$$

Parametric: $x = 3 \cos \theta$, $y = \sin \theta$, $0 \leq \theta \leq 2\pi$, $dy = \cos \theta \, d\theta$
 $dx = -3 \sin \theta \, d\theta$

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} 3 \cos^2 \theta \, d\theta + \frac{1}{2} \int_0^{2\pi} 3 \sin^2 \theta \, d\theta = \frac{3}{2} \int_0^{2\pi} 1 \, d\theta = \boxed{3\pi}$$

7. (15 points) Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 3 & -2 & 4 & 3 \end{bmatrix}.$$

(a) (5 points) Find a basis for the range (column space) of A .Row reduce A , take columns corresponding to leading ones:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 3 & -2 & 4 & 3 \end{bmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{So basis is } \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\} =: \{v^1, v^2\}$$

$$\text{Check: } \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 2v^1 + v^2 \quad \checkmark$$

$$\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = 3v^1 + 3v^2 \quad \checkmark$$

(b) (5 points) Find a basis for the nullspace of A .

From row reduction, want to solve $Ax = 0$, so:

$$x_4 = t, \quad x_3 = s, \quad x_2 = -3t - s,$$

$$x_1 = x_2 - x_3 = -3t - 2s$$

$$\text{So } x = \begin{bmatrix} -3t - 2s \\ -3t - s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{B}_{\text{null}} = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} =: \{w^1, w^2\}.$$

$$\text{Check: } Aw^1 = 0 \quad \checkmark$$

$$Aw^2 = 0 \quad \checkmark$$

(c) (5 points) What are the nullity and rank of A ?

$$\text{Nullity} = \text{dimension of nullspace} = 2$$

$$\text{Rank} = \text{dimension of range} = 2,$$

8. (15 points) Answer the following either TRUE or FALSE. Give short justifications for your answers. Each part is worth 3 points.

(a) Let u and v be two linearly independent vectors in \mathbb{R}^3 . Then there exists a vector $w \in \mathbb{R}^3$ such that $\{u, v, w\}$ is a basis for \mathbb{R}^3 . True

(b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$. The determinant of A is 15. False

(c) If A is as in part 5b then $A^{-1} = \begin{bmatrix} 0 & -\frac{3}{5} & \frac{2}{5} \\ 0 & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{3} & \frac{1}{15} & -\frac{4}{15} \end{bmatrix}$. True

(d) Let A and B be $n \times n$ matrices. If A and B are both nonsingular then AB is nonsingular. True

(e) The vector $u = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. True

(a) From Thm 9, p. 207: can extend u, v to a spanning set.

(b) Cofactor expansion about column 3:

$$\det(A) = (3)(-1)^{1+3} \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = 3(-5) = -15$$

(c) Check: $AA^{-1} = I$ ✓

(d) $(AB)^{-1} = B^{-1}A^{-1}$.

$$(e) \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 2-i \\ 1+2i \end{bmatrix} = (2-i) \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \checkmark$$

Eigenvector is $2-i$.

9. (15 points) Let

$$A = \begin{bmatrix} 6 & 4-2a \\ 4-2a & 3a \end{bmatrix}, \quad u = s \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

for parameters a , s , and t .

- (a) (5 points) Show that u and v are eigenvectors of A for any choice of a .
 (b) (5 points) What is the largest eigenvalue of A when $a = 3$? What is the largest eigenvalue of A when $a = 1$?

$$(a) Au = s \begin{bmatrix} 6 & 4-2a \\ 4-2a & 3a \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = s \begin{bmatrix} 16-2a \\ 8-a \end{bmatrix} = (8-a)s \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So u is an eigenvector with eigenvalue $8-a$

$$Av = t \begin{bmatrix} 6 & 4-2a \\ 4-2a & 3a \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = t \begin{bmatrix} -2+4a \\ 4-8a \end{bmatrix} = (4a-2)t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So v is an ~~eigen~~^{vector} with eigenvalue $4a-2$

(b) The two eigenvalues are $8-a$ and $4a-2$

$a=3$: Eigenvalues are 5 and 10, so largest eigenvalue is 10.
 Check: $Av = 10v$ ✓

$a=1$: Eigenvalues are 7 and 2, so largest eigenvalue is 7.
 Check: $Au = 7u$ ✓