B

Name:

Multivariable Calculus and Matrix Algebra, MATH2010

Second Exam, Wednesday, October 8, 2003.

You may use one sheet of handwritten notes, but no other sources. The exam consists of four questions, and lasts fifty minutes. Please work all three problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed.

Please ring your section below.

7: 11am	6: 1pm	8: 3pm	5: 4pm
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SOLUTIONS

Q1	
Q2	
Q3	
Q4	
Total	
Grade	

1. (25 points) Find the arc length of the plane curve C given by $\mathbf{r}(t) = 2t^{3/2}\mathbf{i} - 4t\mathbf{j}$, $0 \le t \le 1$.

$$\text{Arc length} = \int_C 1\, ds$$

$$= \int_{t=0}^{1} |\underline{r}'(t)|\, dt = \int_0^1 \sqrt{(3t^{\frac{1}{2}})^2 + (4)^2}\, dt$$

$$= \int_0^1 \sqrt{9t+16}\, dt = \frac{1}{9}\cdot\frac{2}{3}\left[(9t+16)^{3/2}\right]_0^1$$

$$= \frac{2}{27}(125-64)$$

$$= \boxed{\frac{122}{27}}$$

2. (25 points) The vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. The curve C is given by $\mathbf{r}(t) = 5\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $0 \leq t \leq \frac{\pi}{2}$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\int_C \underline{F} \cdot d\underline{r} = \int_{t=0}^{\pi/2} \underline{F} \cdot \frac{d\underline{r}}{dt}\, dt$$

$$= \int_{t=0}^{\pi/2} \Big((-2\sin t)(-5\sin t) + (5\cos t)(2\cos t) \Big) \cdot dt$$

$$= 10 \int_{t=0}^{\pi/2} (\sin^2 t + \cos^2 t)\, dt$$

$$= 10 \int_{t=0}^{\pi/2} 1\, dt$$

$$= \boxed{5\pi}$$

3. (25 points) Let $f(x, y, z) = 3xy + 2y^2z + 3xz$, and define the vector field $\mathbf{F} = \nabla f = (3y + 3z)\mathbf{i} + (3x + 4yz)\mathbf{j} + (2y^2 + 3x)\mathbf{k}$.

   (a) (15 points) Let C be the curve given by $\mathbf{r}(t) = \sin(\frac{\pi t}{2})\mathbf{i} + (2^t - 1)\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   (b) (10 points) What is curl $(\mathbf{F})$? Justify your answer.

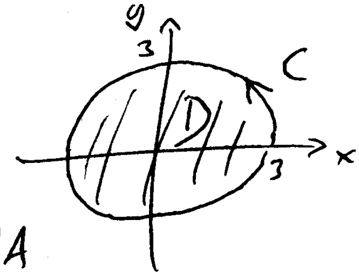
(a) Since $\underline{F} = \nabla f$, we have that $\underline{F}$ is conservative.

Therefore
$$\int_C \underline{F} \cdot d\underline{r} = f(\underline{r}(1)) - f(\underline{r}(0))$$
$$= f(1,1,1) - f(0,0,0)$$
$$= 8 - 0$$
$$= \boxed{8}$$

(b) Since $\underline{F}$ is conservative, curl $(\underline{F}) = \underline{0}$.

4. (25 points) Let C be the circle of radius 3 centered at the origin, traversed in a counterclockwise direction. Show that $\oint_C y^2 dx - x^2 dy = 0$.

Does it follow that $\mathbf{F} = y^2\mathbf{i} - x^2\mathbf{j}$ is conservative? Justify your answer.



Use Green's Theorem:

$$\oint_C y^2 dx - x^2 dy = \iint_D \frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(y^2)\, dA$$

$$= \iint_D (-2x - 2y)\, dA$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3} (-2r\sin\theta - 2r\cos\theta)\, r\, dr\, d\theta \quad \text{in polar coordinates}$$

$$= \int_{\theta=0}^{2\pi} \left[-\frac{2}{3}r^3\right]_0^3 (\sin\theta + \cos\theta)\, d\theta$$

$$= -18\left[-\cos\theta + \sin\theta\right]_0^{2\pi}$$

$$= \boxed{0}$$

It does not follow that $\underline{F}$ is conservative.

Need $\int_C \underline{F}\cdot d\underline{r}$ to be path independent for every C.

Need $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \ \forall x, y$, while here we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x - 2y$.