C Name:

Multivariable Calculus and Matrix Algebra, MATH2010

Third Exam, Wednesday, November 5, 2003.

You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts fifty minutes. Please work all five problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed.

Please ring your section below.

7: 11am	6: 1pm	8: 3pm	5: 4pm
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SOLUTIONS.

Q1	(15 points)
Q2	(15 points)
Q3	(25 points)
Q4	(25 points)
Q5	(20 points)
Total	(100 points)
Grade	

1. (15 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 - cR_3]{R_1 - bR_3} \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - aR_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$

Check: $AA^{-1} = I$ ✓

2. (15 points) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$. Find two vectors x and y in $\mathbb{R}^2$ with $x \neq y$ and $Ax = Ay$.

$Ax = Ay \iff A(x-y) = 0. \iff x-y$ in nullspace of A.

$\iff x - y = t\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ for some t.

So, eg, $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $y = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

Then $Ax = \begin{bmatrix} 10 \\ 10 \end{bmatrix} = Ay.$

3. (25 points) Let W be the subspace of $\mathbb{R}^3$ spanned by the three vectors u, v, and w given below:

$$u := \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \qquad v := \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}, \qquad w := \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

(a) (9 points) Show that the three vectors are linearly dependent.

(b) (8 points) Find a basis for W.

(c) (8 points) Express v as a linear combination of the vectors in your basis.

(a) $ux_1 + vx_2 + wx_3 = 0$:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ -2 & -3 & 3 & 0 \\ -3 & -5 & 4 & 0 \end{bmatrix} \xrightarrow[r_3+3r_1]{r_2+2r_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[r_3-r_2]{r_1-2r_2} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x = t\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, so nontrivial x exist, so linearly dependent.

(b) Can take the vectors corresponding to the leading ones.

So: u, v give basis.

(c) $v = 0u + 1v$

4. (25 points) Let the four vectors u, v, w, and z in $\mathbb{R}^4$ be defined below.

$$u = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 2 \end{bmatrix}, \qquad w = \begin{bmatrix} 5 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \qquad z = \begin{bmatrix} 8 \\ -2 \\ -1 \\ 6 \end{bmatrix}.$$

The vectors u and v form a basis for the row space of a certain 3×4 matrix A. (Be sure to justify your answers below.)

(a) (5 points) What is the rank of A?

(b) (5 points) What is the nullity of A?

(c) (5 points) Is w in the null space of A?

(d) (5 points) Is z in the null space of A?

(e) (5 points) Do the vectors w and z form a basis for the null space of A?

(a) 2 vectors in basis for row space, so rank = $\boxed{2}$

(b)

See solutions to Version A.

5. (20 points) Let $A = \begin{bmatrix} 1 & 4 \\ 4 & a \end{bmatrix}$.

(a) (5 points) For what value of a is A not invertible?

(b) (7 points) Let a^* be the value found in part 5a. Find a vector b so that the system of equations $Ax = b$ is inconsistent when $a = a^*$.

(c) (8 points) Let a^* be the value found in part 5a. Now let $a = a^* + \epsilon$, where ϵ is a small positive number. Find the solution to the system of equations $Ax = b$, with b taken to be the vector you found in part 5b. Show that the norm of the solution vector x gets larger as ϵ gets closer to zero.

(a) If A is invertible: $A^{-1} = \frac{1}{a-16}\begin{bmatrix} a & -4 \\ -4 & 1 \end{bmatrix}$

So not invertible if $\boxed{a = 16}$

(b) Want b so $\begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is inconsistent.

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 4 & 16 & b_2 \end{array}\right] \xrightarrow{R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 0 & b_2 - 4b_1 \end{array}\right]$$

So need $b_2 \neq 4b_1$. Eg: $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 4 \\ 4 & 16+\varepsilon \end{bmatrix}$ $A^{-1} = \frac{1}{\varepsilon}\begin{bmatrix} 16+\varepsilon & -4 \\ -4 & 1 \end{bmatrix}$

Soln to $Ax = b$: $x = A^{-1}b = \frac{1}{\varepsilon}\begin{bmatrix} 16+\varepsilon & -4 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\varepsilon}\begin{bmatrix} 16+\varepsilon \\ -4 \end{bmatrix}$

$\|x\| = \sqrt{x^T x} = \frac{1}{\varepsilon}\sqrt{((16+\varepsilon)^2 + 16)} \approx \frac{\sqrt{272}}{\varepsilon}$ for ε small

$\to \infty$ as $\varepsilon \to 0$.