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Name:

Multivariable Calculus and Matrix Algebra, MATH2010

First Exam, Friday, September 19, 2003.

You may use one sheet of handwritten notes, but no other sources. The exam consists of three questions, and lasts fifty minutes. Please work all three problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed.

Please ring your section below.

7: 11am
Xiaoyun
Ji

6: 1pm
Shawne
Campbell

8: 3pm
Shawne
Campbell

5: 4pm
Xiaoyun
Ji

Solutions.

Q1	
Q2	
Q3	
Total	
Grade	

1. (40 points) Let $z = x^2 + y^2 + 2xy$, with $x = r \cos(\theta)$ and $y = r \sin(\theta)$.

(a) (20 points) What is $\frac{\partial z}{\partial r}$? Evaluate this quantity at the point $(x, y) = (0, 4)$.

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} \\ &= (2x + 2y) \cos \theta + (2y + 2x) \sin \theta \\ &= 2r \cos^2 \theta + 2r \sin \theta \cos \theta + 2r \sin^2 \theta + 2r \sin \theta \cos \theta \\ &= \boxed{2r + 4r \sin \theta \cos \theta}\end{aligned}$$

$$\text{At } (x, y) = (0, 4): \quad \theta = \frac{\pi}{2}, \quad r = 4$$

$$\text{So } \frac{\partial z}{\partial r} = 8 \quad \text{since } \cos\left(\frac{\pi}{2}\right) = 0.$$

(b) (20 points)

Let D be the quarter circle given by $x^2 + y^2 \leq 4$, $x \geq 0$, $y \geq 0$. Find the volume under the surface $z = x^2 + y^2 + 2xy$ and above the region D ,

$$\iint_D z dA.$$

(Hint: $\int \sin \theta \cos \theta d\theta = -\frac{1}{4} \cos(2\theta)$.)

$$\iint_D z dA = \int_{\theta=0}^{\pi/2} \int_{r=0}^2 (r^2 + 2r^2 \sin \theta \cos \theta) r dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4} + \frac{r^4}{2} \sin \theta \cos \theta \right]_0^2 d\theta$$

$$= \int_{\theta=0}^{\pi/2} (4 + 8 \sin \theta \cos \theta) d\theta$$

$$= \left[4\theta - 2 \cos(2\theta) \right]_0^{\pi/2}$$

$$= \boxed{2\pi + 4}$$

2. (30 points) Define

$$f(x, y, z) = xyz$$

$$g(x, y, z) = x - 5y - 2z + 5$$

$$h(x, y, z) = 3x + 3y + 2z - 19.$$

We want to minimize $f(x, y, z)$ subject to satisfying the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Show that the point $(x, y, z) = (4, 1, 2)$ satisfies the Lagrange system of equations.

First check: $g(4, 1, 2) = 0 \checkmark$ $h(4, 1, 2) = 0 \checkmark$

Now find λ, μ so that $\nabla f = \lambda \nabla g + \mu \nabla h$:

$$1\lambda + 3\mu = yz = 2 \quad (1)$$

$$-5\lambda + 3\mu = xz = 8 \quad (2)$$

$$-2\lambda + 2\mu = xy = 4 \quad (3)$$

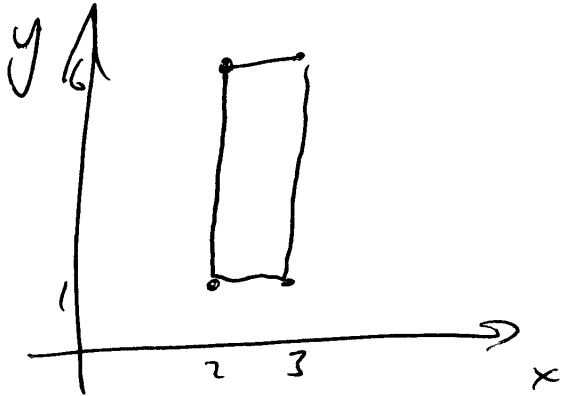
$$(1) - (2) \Rightarrow \lambda = -1 \Rightarrow \boxed{\lambda = -1}$$

$$\text{Substitute in (1): } -1 + 3\mu = 2 \Rightarrow \boxed{\mu = 1}$$

$$\text{Must check (3): } -2\lambda + 2\mu = 2 + 2 = 4 \checkmark$$

$$\text{Should check (2): } -5\lambda + 3\mu = 5 + 3 = 8 \checkmark$$

3. (30 points) A rectangular lamina has corners $(2,1)$, $(2,6)$, $(3,1)$, $(3,6)$. It has density $\rho(x,y) = ye^{xy}$. Find its mass. (Note: It is OK to leave the answer as a sum of exponential functions.)



$$\begin{aligned}
 \text{Mass } m &= \int_1^6 \int_2^3 ye^{xy} dx dy && \text{(Note order of integration)} \\
 &= \int_1^6 [e^{xy}]_2^3 dy \\
 &= \int_1^6 (e^{3y} - e^{2y}) dy \\
 &= \left[\frac{1}{3} e^{3y} - \frac{1}{2} e^{2y} \right]_1^6 \\
 &= \frac{1}{3} e^{18} - \frac{1}{2} e^{12} - \frac{1}{3} e^3 + \frac{1}{2} e^2
 \end{aligned}$$