A

Name:

Multivariable Calculus and Matrix Algebra, MATH2010

First Exam, Friday, September 19, 2003.

You may use one sheet of handwritten notes, but no other sources. The exam consists of three questions, and lasts fifty minutes. Please work all three problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed.

Please ring your section below.

7: 11am	6: 1pm	8: 3pm	5: 4pm
Xiaoyun Ji	Shawne Campbell	Shawne Campbell	Xiaoyun Ji

SOLUTIONS.

Q1	
Q2	
Q3	
Total	
Grade	

1. (40 points) Let $z = x^2 + y^2 + 4xy$, with $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

   (a) (20 points) What is $\frac{\partial z}{\partial r}$? Evaluate this quantity at the point $(x, y) = (3, 0)$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= (2x+4y)\cos\theta + (2y+4x)\sin\theta$$

$$= (2r\cos\theta + 4r\sin\theta)\cos\theta + (2r\sin\theta + 4r\cos\theta)\sin\theta$$

$$= 2r\cos^2\theta + 2r\sin^2\theta + 8r\sin\theta\cos\theta$$

$$= 2r + 8r\sin\theta\cos\theta$$

At $(x,y) = (3,0)$: $r=3,\ \theta=0$

So $\frac{\partial z}{\partial r} = 2r = 6$ since $\sin\theta = 0$.

(b) (20 points)

Let D be the quarter circle given by $x^2 + y^2 \leq 4$, $x \geq 0$, $y \geq 0$. Find the volume under the surface $z = x^2 + y^2 + 4xy$ and above the region D,

$$\int\int_D z dA.$$

Hint: $\int \sin\theta \cos\theta \, d\theta = -\frac{1}{4}\cos 2$

$$\iint_D z\,dA = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\left(r^2 + 4r^2\sin\theta\cos\theta\right)r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4} + r^4\sin\theta\cos\theta\right]_0^2 d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left(4 + 16\sin\theta\cos\theta\right)d\theta$$

$$= \left[4\theta - 4\cos 2\theta\right]_0^{\pi/2}$$

$$= 4\frac{\pi}{2} + 4 + 4 = 2\pi + 8$$

2. (30 points) Define

$$\begin{aligned} f(x,y,z) &= xyz \\ g(x,y,z) &= 3x+2y+3z-19 \\ h(x,y,z) &= -5x-2y+z+5. \end{aligned}$$

We want to minimize $f(x,y,z)$ subject to satisfying the constraints $g(x,y,z)=0$ and $h(x,y,z)=0$. Show that the point $(x,y,z)=(1,2,4)$ satisfies the Lagrange system of equations.

Want $\nabla f = \lambda \nabla g + \mu \nabla h$

That is, $yz = 3\lambda - 5\mu$

$xz = 2\lambda - \cancel{5}2\mu$

$xy = 3\lambda + \mu$

Note that $(1,2,4)$ satisfies $g(x,y,z)=0$, $h(x,y,z)=0$ ✓

Also, equations become:

$8 = 3\lambda - 5\mu$ (1)

$4 = 2\lambda - 2\mu$ (2)

$2 = 3\lambda + \mu$ (3)

(1) − (3) $\Rightarrow 6 = -6\mu \Rightarrow \boxed{\mu = -1.}$

Plug in (1): $3\lambda = 8 + 5\mu = 3 \Rightarrow \boxed{\lambda = 1}$

Need to check (2) is consistent:

$2\lambda - 2\mu = 2 + 2 = 4$ ✓

Also (3): $3\lambda + \mu = 3 - 1 = 2$ ✓.

So we do satisfy the Lagrange equations, with multipliers $\lambda = 1, \mu = -1$.

3. (30 points) A square lamina has corners (1,2), (1,5), (3,2), (3,5). It has density $\rho(x, y) = ye^{xy}$. Find its mass. (Note: It is OK to leave the answer as a sum of exponential functions.)

Mass $m = \iint_D \rho(x,y)\, dA$

$$= \int_1^3 \int_2^5 ye^{xy}\, dy\, dx$$

$$= \int_2^5 \int_1^3 ye^{xy}\, dx\, dy \quad \text{change order of integration}$$

$$= \int_2^5 \left[e^{xy} \right]_1^3 dy$$

$$= \int_2^5 \left[e^{3y} - e^{y} \right] dy$$

$$= \left[\tfrac{1}{3} e^{3y} - e^{y} \right]_2^5$$

$$= \boxed{\tfrac{1}{3} e^{15} - e^{5} - \tfrac{1}{3} e^{6} + e^{2}}$$