INTRODUCTION TO DIFFERENTIAL EQUATIONS, TEST 2
Sections 13-16, Fall 2002

Section meeting time ____________ Name ______________________________

Instructions. You are allowed to use one 8 1/2 × 11 inch sheet of paper of notes. No calculators, PDAs, computers, books, or cellular phones are allowed. Do not collaborate in any way. In order to receive credit, your answers must be clear, legible, and coherent.

1. (5 points each) Fill in the blank with the letter corresponding to the best description. Use
A = simple harmonic motion; B = overdamped; C = critically damped; D = underdamped; E = resonance ; F = beating; G = transient plus harmonic steady-state.

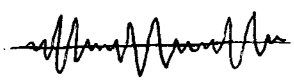
i) __A__ $\ddot{u} + 9u = 0$ Try $u = e^{rt}$ $r^2 + 9 = 0 \Rightarrow r = 3i$
$u = A\cos 3t + B\sin 3t$

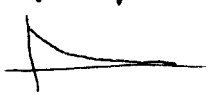
ii) __E__ $\ddot{u} + 9u = \cos 3t$

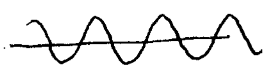
iii) __F__ $\ddot{u} + 9u = \cos t$

iv) __D__ $\ddot{u} + 2\dot{u} + 9u = 0$ $r^2 + 2r + 9 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-36}}{2}$

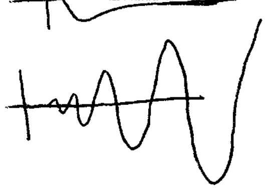
v) __G__ $\ddot{u} + 2\dot{u} + 9u = \cos t$

vi) __F__ 

vii) __B__ 

viii) __A__ 

ix) __C__ 

x) __E__ 

2. (25 points) Find all nonzero solutions and the corresponding values of λ for the two-point boundary-value problem $X'' + 2X' + \lambda X = 0$; (Hint: consider the cases $\lambda = 1$, $\lambda > 1$, and $\lambda < 1$.

$X(0) = 0 = X(1)$

Case $\lambda = 1$ $\quad X'' + 2X' + X = 0 \quad$ Try $X = e^{rx} \quad r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$

$X(x) = Ae^{-x} + Bxe^{-x}$

$0 = X(0) = A$

$0 = X(1) = Be^{-1} \Rightarrow B = 0$

$X =$ no nontrivial solutions

Case $\lambda > 1$ $\quad r^2 + 2r + \lambda = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{4 - 4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}$

$= -1 \pm \sqrt{\lambda - 1}\, i$

$X(x) = e^{-x}\left(A\cos\left(\sqrt{\lambda-1}\,x\right) + B\sin\sqrt{\lambda-1}\,x\right)$

$0 = X(0) = A$

$0 = X(1) = e^{-1}B\sin\sqrt{\lambda-1} \Rightarrow \sqrt{\lambda-1} = n\pi \Rightarrow \lambda - 1 = (n\pi)^2$

$\lambda = 1 + (n\pi)^2$; $X = e^{-x}\sin n\pi x$

Case $\lambda < 1$

$X(x) = Ae^{(-1+\sqrt{1-\lambda})x} + Be^{(-1-\sqrt{1-\lambda})x} = e^{-1}\left(a\cosh\sqrt{1-\lambda}\,x + b\sinh\sqrt{1-\lambda}\,x\right)$

$0 = X(0) = e^{-1}a$

$0 = X(1) = e^{-1}b\sinh\sqrt{1-\lambda} \Rightarrow \sqrt{1-\lambda} = 0 \Rightarrow \lambda = 1 \Rightarrow\Leftarrow$

$\lambda =$; $X =$ no nontrivial solutions

3. (25 points) Solve

$$u_t = u_{xx}$$

$$u(0,t) = 0 \qquad u(\pi,t) = \pi$$

$$u(x,0) = x + 3\sin 2x + 5\sin 10x.$$

You may use the fact that the problem $u_t = u_{xx}, u(0,t) = 0 = u(L,t)$ has solutions of the form $u_n(x,t) = \sin(n\pi x/L)\exp(-[n\pi/L]^2 t)$.

$u(x,t) = v(x) + w(x,t)$

$V_{xx} = 0 \Rightarrow v = Ax + B$

$0 = v(0) = B$

$\pi = v(\pi) = A\pi \Rightarrow A = 1$

$v = x$

$w_t = w_{xx}$

$w(0,t) = 0 = w(\pi,t)$

$w(x,0) = u(x,0) - v(x)$

$= 3\sin 2x + 5\sin 10x$

$L = \pi \Rightarrow \quad w(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-n^2 t}$

$3\sin 2x + 5\sin 10x = w(x,0) = \sum_{n=1}^{\infty} b_n \sin nx \Rightarrow b_2 = 3$

$b_{10} = 5$

all other b's $= 0$.

$w(x,t) = 3\sin 2x\, e^{-4t} + 5\sin 10x\, e^{-100t}$

$u(x,t) = \underline{x + 3\sin 2x\, e^{-4t} + 5\sin 10x\, e^{-100t}}$