Birge & Louveaux, Page 101, Question 2

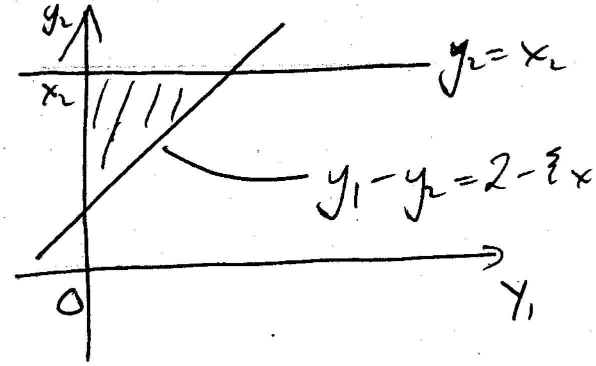
Need to satisfy $y_1 - y_2 \le 2 - \xi x_1$ — ξ is restricted to be nonnegative.

$$y_2 \le x_2$$
$$y_1, y_2 \ge 0.$$

Note that ① if $x_2 < 0$ then no feasible y exists.

② if $x_1 \le 0$ and $x_2 \ge 0$ then $y_1 = 0, y_2 = 0$ is feasible.

If $x_1 \ge 0$, $x_2 \ge 0$:



In both distributions, $\xi \geq 0$.

As ξ and/or x_1 increases, the diagonal line moves to the left.

So the problem is feasible if $y_1 = 0, y_2 = x_2$ is feasible,

that is, if $-x_2 \le 2 - \xi x_1$

So $\underline{K_2(\xi) = \{x : \xi x_1 - x_2 \le 2, \ x_2 \ge 0\}}$

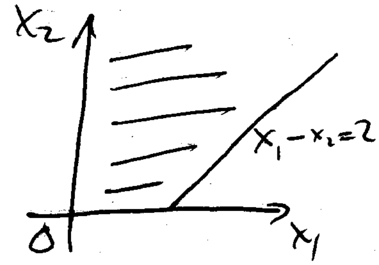
page 101, Q2

(a) $\xi \in \mathcal{U}[0,1]$.

$$K_2 = \bigcap_{\xi \in [0,1]} K_2(\xi)$$

$$= \bigcap_{\xi \in [0,1]} \left\{x : x_1 \leq \frac{2+x_2}{\xi},\ x_2 \geq 0\right\}$$

$$= \underline{\underline{\{x : x_1 - x_2 \leq 2,\ x_2 \geq 0\}}}$$



(b) $\xi \sim \text{Poisson}(\lambda),\ \lambda > 0$

$$K_2 = \bigcap_{\xi} \left\{x : x_1 \leq \frac{2+x_2}{\xi},\ x_2 \geq 0\right\}$$

$$= \underline{\underline{\{x : x_1 \leq 0,\ x_2 \geq 0\}}}$$

since $K_2(\xi)$ restrict x_1 more and more as ξ grows.

In both cases, W is fixed and ξ has finite second moments, so $K_2 = K_2^P$ and K_2 is closed and convex.

In (a), the support Ξ is ~~finite~~, polyhedral, so K_2 is polyhedral.

Birge & Louveaux, Page 101, Q3

$$\mathcal{Q}(x) = \int_1^\infty Q(x,\xi) f(\xi)\, d\xi$$

$Q(x,\xi) = \max\{\xi, x\}$, finite for all ξ.

So $K_2^P = \{x : x \geq 0\}$.

$$\mathcal{Q}(x) = \int_1^x x f(\xi) d\xi + \int_x^\infty \xi f(\xi) d\xi \qquad \text{if } x \geq 1$$

$$= \int_1^x \frac{2x}{\xi^3} d\xi + \int_x^\infty \frac{2}{\xi^2} d\xi$$

$$= \left[-x/\xi^2 \right]_1^x - \left[2/\xi \right]_x^\infty$$

$$= x - \frac{1}{x} + 2/x = x + \frac{1}{x} \qquad \text{if } x \geq 1.$$

If $x < 1$:
$$\mathcal{Q}(x) = \int_1^\infty \xi f(\xi) d\xi = \int_1^\infty 2/\xi^2 d\xi = \left[-2/\xi \right]_1^\infty$$

$$= 2.$$

So $\mathcal{Q}(x)$ is finite for all x, so $K_2 = K_2^P = \{x : x \geq 0\}$.

ξ does not have bounded second moments:

$$\int_1^\infty \xi^2 f(\xi) d\xi = \int_1^\infty 2/\xi \, d\xi, \quad \text{divergent.}$$

Birge & Louveaux, Page 102, Question 4.

(a) Write constraints as equality constraints:

$$
\begin{aligned}
\min \quad & 2y_1 + y_2 \\
\text{s.t.} \quad & y_1 + y_2 - y_3 = 1 - x_1 \\
& y_1 - y_4 = \xi - x_1 - x_2 \\
& y_1, y_2, y_3, y_4 \geq 0.
\end{aligned}
$$

So $W = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

To show complete recourse, need to verify that $Wy = t, y \geq 0$ has a solution for every t.

Let $y_1 = \max\{0, t_1, t_2\}$, $y_2 = 0$, $y_3 = y_1 - t_1$, $y_4 = y_1 - t_2$.

This satisfies $Wy = t, y \geq 0$.

So the problem has complete recourse.

Note that the information about $E(\xi)$ is irrelevant.

page 102, Question 4

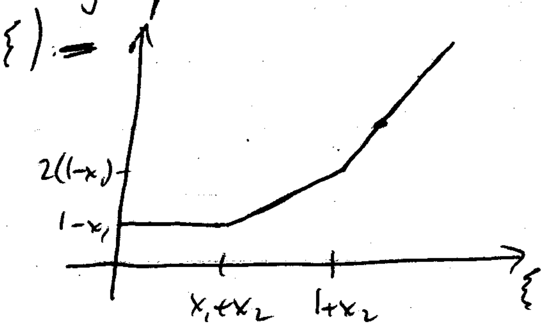
(b) min $2y_1 + y_2$ (P)
s.t. $y_1 + y_2 \geq 1 - x_1$
$y_1 \geq \xi - x_1 - x_2$
$y_1, y_2 \geq 0$

Dual is: max $(1-x_1)z_1 + (\xi - x_1 - x_2)z_2$ (D)
s.t. $z_1 + z_2 \leq 2$
$z_1 \leq 1$
$z_1, z_2 \geq 0.$

If $\xi \geq x_1 + x_2$: Given soln y has value $2(\xi - x_1 - x_2) + (1 - \xi + x_2)^+$
If $(1 - \xi + x_2) \leq 0$:
Let $z_1 = 0$, $z_2 = 2$.
Then primal & dual feasible, and optimal values are both $2(\xi - x_1 - x_2)$.
If $(1 - \xi + x_2) > 0$:
Let $z_1 = 1$, $z_2 = 1$.
Then primal & dual feasible, and optimal values are both $1 + \xi - 2x_1 - x_2$

If $\xi \leq x_1 + x_2$:
Let $z_1 = 1$, $z_2 = 0$
Then primal and dual feasible, and optimal values are both $1 - x_1$.

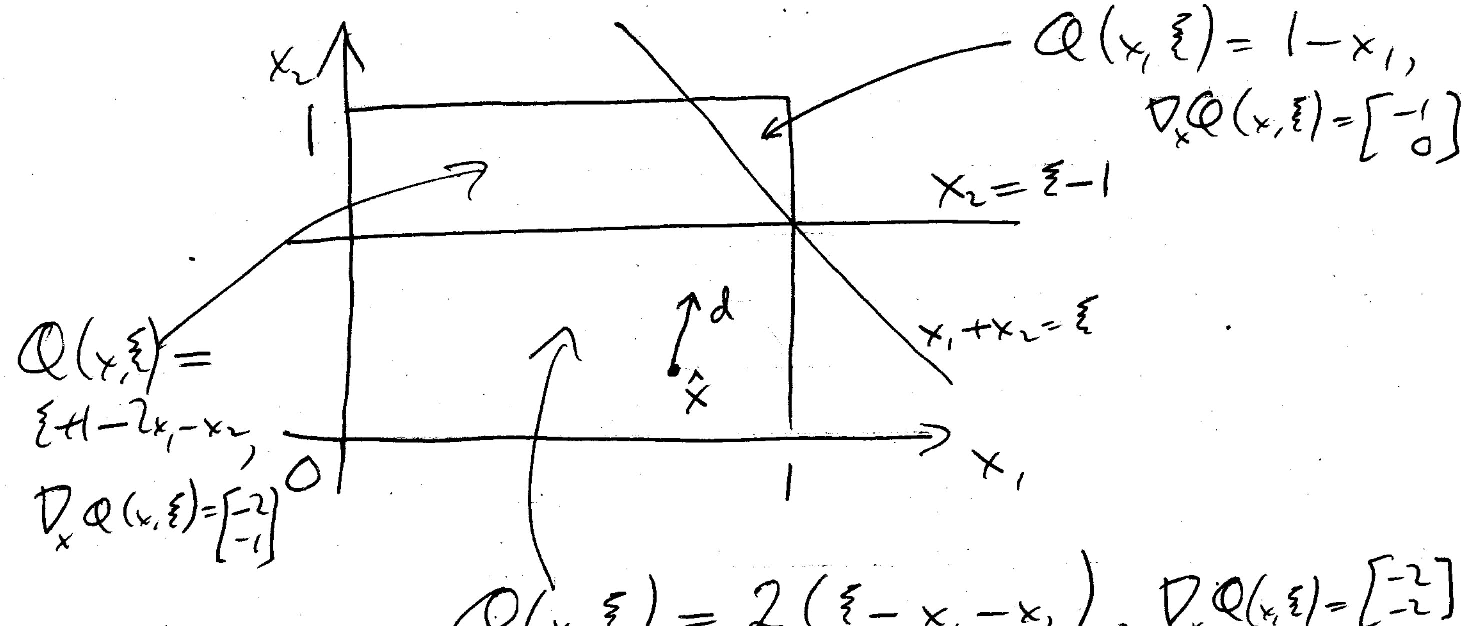
From the above, we immediately get $Q(x, \xi)$ as stated.
$Q(x, \xi)$ is clearly piecewise linear in either ξ or x_1 and x_2



Fix x_1
vary ξ

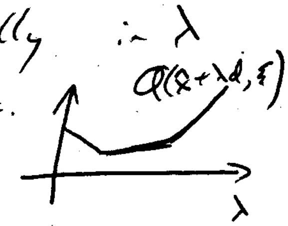
page 102, Question 4

If we fix ξ and vary x_1, x_2, show $Q(x, \xi)$ is convex and piecewise linear:



$Q(x, \xi)$ is linear on each segment.
It is continuous across the boundaries of the segments.

For any $\hat{x}$ and any direction d, the directional derivative ~~changes monotonically along the line~~ ~~changes~~ increases monotonically in λ for points $\hat{x} + \lambda d$. So function is convex.



Eg: $\xi = 1.1$, $\hat{x} = (1, 0)$, $d = (-2, 3)$:

$$d^T \nabla_x Q(\hat{x} + \lambda d, \xi) = \begin{cases} -6 & \text{if } 0 \le \lambda < \frac{1}{50} \\ -1 & \text{if } \frac{1}{50} < \lambda < \frac{1}{30} \\ 2 & \text{if } \frac{1}{30} < \lambda < \frac{1}{5} \end{cases}$$

page 102, Question 4

(c) $\xi \sim U[0,2]$.

$$Q(x) = \tfrac{1}{4}(x_1^2 + 2x_2^2 + 2x_1x_2 - 8x_1 - 6x_2 + 9)$$

We have fixed recourse and finite second moments for ξ.

Note that this expression for $Q(x)$ does not apply for all x. It is valid within $0 \le x_1 \le 1, 0 \le x_2 \le 1$.

As written, $Q(x)$ is not Lipschitzian everywhere.

If $x_1 \geq 1$ and $x_1 + x_2 \geq 2$, then $Q(x, \xi) = 0$ for all $\xi \in [0,2]$, so $Q(x) = 0$, for example.

So we only check Thm 6 for $0 \le x_1 \le 1$, $0 \le x_2 \le 1$.

$Q(x)$ is convex: $\nabla Q(x) = \begin{bmatrix} \frac{1}{2}x_1 + \frac{1}{2}x_2 - 8 \\ \frac{1}{2}x_1 + x_2 - 6 \end{bmatrix}$

Hessian: $\nabla^2 Q(x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

$\nabla^2 Q(x)$ is positive definite, so $Q(x)$ is convex.
$Q(x)$ is differentiable.
Since we're only checking a ~~finite~~ bounded domain, and since Q is differentiable, it is Lipschitzian, from the Mean Value Theorem.

Birge & Louveaux, Page 102, Question 5

If $\xi < 0$: $\Delta y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a ray that drives the objective to $-\infty$, so $Q(x,\xi) = -\infty$

If $0 \leq \xi \leq 1$: Optimal y is $y_1 = 1 - x_1$, $y_2 = 0$.

$$Q(x,\xi) = \xi(1-x_1)$$

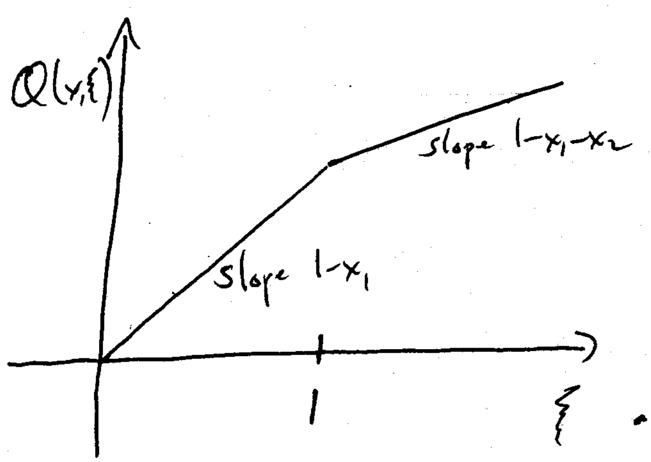
If $\xi \geq 1$: Ⓐ If $x_1 + x_2 \geq 1$: Optimal y is $y_1 = 0$, $y_2 = 1 - x_1$

So $Q(x,\xi) = 1 - x_1$

Ⓑ If $x_1 + x_2 \leq 1$: Optimal y is $y_1 = 1 - x_1 - x_2$, $y_2 = x_2$

So $Q(x,\xi) = \xi(1-x_1) + (1-\xi)x_2$

$x_1 + x_2 \leq 1$:



$x_1 + x_2 \geq 1$: