4800 - Numerical Computing

Test 2 - November 12, 2001
Time: 10:00 - 11:50

This test contains FOUR pages (including this page) and SIX questions.

Make sure your copy is complete.

Total number of marks : 100

To obtain full marks you should answer all questions.
1. \[25\text{ marks}\]

A function \(f(x)\) has the values shown below:

\[
\begin{array}{ccccc}
    x & 1 & 1.25 & 1.5 & 1.75 & 2 \\
    f(x) & 10 & 8 & 7 & 6 & 5 \\
\end{array}
\]

In each of your answers to parts (a)-(g), show your working.

(a) Use the composite trapezoidal rule and the function values at \(x = 1, 1.5, 2\) to approximate \(\int_1^2 f(x)\,dx\).

(b) Use the composite trapezoidal rule and all the function values to approximate \(\int_1^2 f(x)\,dx\).

(c) Use Romberg integration and the results from parts (a) and (b) to approximate \(\int_1^2 f(x)\,dx\).

(d) Use Simpson’s rule and the function values at \(x = 1, 1.5, 2\) to approximate \(\int_1^2 f(x)\,dx\).

(e) Use the composite Simpson’s rule and all the function values to approximate \(\int_1^2 f(x)\,dx\).

(f) Derive an integration rule that is more accurate than Simpson’s rule and is obtained from Simpson’s rule in a manner that is analogous to the derivation of Romberg integration from the trapezoidal rule.

(g) Using the results from parts (d) and (e) and the rule derived in (f), obtain an approximation to \(\int_1^2 f(x)\,dx\).

2. \[20\text{ marks}\]

Derive the 3-point Gaussian quadrature rule of the form

\[
\int_{-1}^{1} f(x) \, dx \approx af(-\alpha) + bf(0) + cf(\alpha)
\]

that has the highest possible precision. That is, determine the values for \(a, b, c, \alpha\). What is the precision of this integration rule?
3. [10 marks]

We wish to obtain an approximation for

\[ \int_{0}^{\pi/2} \cos x \, dx. \]

In each of the following cases and without using either method, obtain values for the number of subintervals that must be used to obtain an approximation with error not exceeding \(2 \times 10^{-4}\);

(a) composite trapezoidal rule

(b) composite Simpson’s rule

Show your working.

4. [13 marks]

List all of the ways in which each of the following functions fails to be a natural cubic spline.

(a)

\[ S(x) = \begin{cases} 
  x + 1 & -2 \leq x \leq -1 \\
  x^3 - 2x - 1 & -1 \leq x \leq 1 \\
  x - 3 & 1 \leq x \leq 2
\]\

(b)

\[ S(x) = \begin{cases} 
  x^3 + x - 1 & -1 \leq x \leq 0 \\
  x^3 - x - 1 & 0 \leq x \leq 1
\]
5. [23 marks]

(a) Derive the quadratic polynomial in Lagrange form that interpolates $f(x) \equiv 1/x$ at $x = 2, 2.5$ and 4.

(b) Use this polynomial to obtain an approximation to $f(\frac{1}{3})$.

(c) Compare the actual error in this approximation to $f(\frac{1}{3})$ with the error estimate obtained using the error term for quadratic interpolation.

6. [9 marks]

Suppose you are given the following data from a function $f(x)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-2</td>
<td>6</td>
<td>-4</td>
<td>-1</td>
</tr>
</tbody>
</table>

In the questions below you are to choose from the following methods to interpolate these points:

- Lagrange interpolation
- piecewise linear interpolation
- cubic Hermite interpolation
- cubic spline interpolation

Without using any of the above methods, answer the following. In each case give reason(s) for your answer.

(a) Which method will produce the smoothest interpolation function over the interval?

(b) Which method will produce an interpolation function that can be evaluated fastest?

(c) Which method cannot be used because you are not given enough data?