Rotation of Axes

\[ x = r \cos \theta, \quad y = r \sin \theta \quad \text{Polar Coordinates} \]

Let \[ x' = r \cos(\theta + \alpha), \quad y' = r \sin(\theta + \alpha) \]

Then
\[ x' = r \cos \theta \cos \alpha - r \sin \theta \sin \alpha, \]
\[ y' = r \sin \theta \cos \alpha + r \cos \theta \sin \alpha. \]

Using the polar forms for \( x \) and \( y \)
\[ x' = x \cos \alpha - y \sin \alpha \]
\[ y' = y \cos \alpha + x \sin \alpha. \]

Now solve for \( x \) and \( y \) in terms of \( x' \) and \( y' \). To find \( x \), multiply the equation for \( x' \) by \( \cos \alpha \), and the equation for \( y' \) by \( \sin \alpha \):
\[ x' \cos \alpha = x \cos^2 \alpha - y \sin \alpha \cos \alpha \]
\[ y' \sin \alpha = y \cos \alpha \sin \alpha + x \sin^2 \alpha. \]

Add the last two equations and since \( \cos^2 \alpha + \sin^2 \alpha = 1 \),
\[ x' \cos \alpha + y' \sin \alpha = x. \]

Similarly, to find \( y \), multiply the equation for \( x' \) by \( -\sin \alpha \) and the equation for \( y' \) by \( \cos \alpha \)
\[ -x' \sin \alpha = -x \cos \alpha \sin \alpha + y \sin^2 \alpha \]
\[ y' \cos \alpha = y \cos^2 \alpha + x \sin \alpha \cos \alpha. \]

Add these two equations to get
\[ -x' \sin \alpha + y' \cos \alpha = y(\sin^2 \alpha + \cos^2 \alpha) = y. \]

In summary
\[ x = x' \cos \alpha + y' \sin \alpha \]
\[ y = -x' \sin \alpha + y' \cos \alpha. \]
So, a quadric surface given by

\[ z = Ax^2 + By^2, \]

written in the rotated coordinates becomes

\[ z = A(x' \cos \alpha + y' \sin \alpha)^2 + B(-x' \sin \alpha + y' \cos \alpha)^2 \]
\[ = (A \cos^2 \alpha + B \sin^2 \alpha)x'^2 + 2(A - B)(\cos \alpha \sin \alpha)x'y' + (A \sin^2 \alpha + B \cos^2 \alpha)y'^2. \]

This looks a bit complicated but it is rather general. Consider the case of rotation through an angle of 30° or \( \pi/6 \). Then \( \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} \) and \( \sin \left( \frac{\pi}{6} \right) = \frac{1}{2} \). So

\[ x = \frac{\sqrt{3}}{2} x' + \frac{1}{2} y' \]
\[ y = -\frac{1}{2} x' + \frac{\sqrt{3}}{2} y'. \]

The quadric surface

\[ z = 2x^2 + y^2, \]

which is an \textit{elliptic paraboloid} becomes

\[ z = 2 \left( \frac{\sqrt{3}}{2} x' + \frac{1}{2} y' \right)^2 + \left( -\frac{1}{2} x' + \frac{\sqrt{3}}{2} y' \right)^2 \]
\[ = \frac{7}{4} x'^2 + \frac{\sqrt{3}}{2} x'y' + \frac{5}{4} y'^2. \]

Usually, this technique of rotating axes is used to eliminate the \( xy \)-term.