This assignment is in two parts. The answers to questions in Part I are generally in the books. It is advisable to make every effort to solve the problem before consulting the answer. Page numbers are to the Text, Introduction to Linear Algebra, 5th edition, by Johnson, Riess and Arnold. This, the last assignment, should be turned in promptly.

Part I

1. (a) Text: p.297, #9. (b) p.314, #17.
2. (a) Text: p.314, #3. (b) p.315, #27(c).
3. (a) Text: p.315, #28. Verify that the eigenvectors of the matrix of problem 2 (b) satisfy \( u^T v = \theta \).
   (b) (MAPLE) Text: p.325, #33. Method: Use the Maple command \texttt{eigenvectors(A)}. Put a decimal point somewhere in the specification of \( A \) so that the program will return the answer in decimals, instead of radicals.
4. (MAPLE) Text: p.335. Complete Example 7. Method: Find the eigenvectors of \( A \). Use the Maple command \texttt{GramSchmidt}\{\( w_1, w_2, w_3 \)\}; to produce the orthogonal vectors \( x_1, x_2, x_3 \). (Note that the Maple routine may give a different set from that given in the textbook.) Do the normalization (e.g., by using a command like \texttt{sqrt(dotprod(x_1, x_1))}; to find the lengths of the vectors). Hence produce the desired orthogonal matrix \( Q \) that diagonalizes \( A \). Verify that \( Q^T AQ = D \).

Part II

5. (a) Text: p.314, #16. Show that the matrix has fewer than three linearly independent eigenvectors. Write its characteristic polynomial in factored form, and by substituting in, show that \( A \) satisfies the Cayley-Hamilton theorem. (See the statement on p.306 just before problems 20-23.)
   (b) Find a basis for \( R^3 \) from the eigenvectors of the matrix
   \[
   B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}.
   \]
6. (a) Text: p.325, #41.
   (b) Also prove the converse, that if \( A \) is symmetric with all positive eigenvalues, then it is positive definite. \textit{Hint:} Use p.315 eqn.(12). This completely characterizes symmetric positive definite matrices.
   (c) Determine whether or not the matrix
   \[
   A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}
   \]
   is positive definite. \textit{Note:} There is a command \texttt{definite} in Maple, but you don’t have to use it to solve this problem.
7. (a) Use Theorems 22 (Schur’s theorem) and 18 of chapter 4 to conclude that for any \((n \times n)\) matrix \(A\), with only real eigenvalues, the trace of \(A\) (see Assignment #8), \(\text{tr}(A) = \sum_{i=1}^{n} \lambda_i\), and \(\det(A) = \prod_{i=1}^{n} \lambda_i\), the product of the eigenvalues. (Schur’s theorem and these results are also true for matrices with complex eigenvalues, but you don’t have to prove it.)

(b) Verify that the matrix \(A\) of problem 6 (c) satisfies the results of part (a).

8. Consider the process of information transmission described on the course website (http://eaton.math.rpi.edu/CourseMaterials/Fall01/IIH2010/notes/notes.html) as

\[ x_k = Ax_{k-1}, \quad k = 1, 2, 3, \ldots, \]

where

\[ A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \]

(The sequence \(x_k\) was previously considered in problem 6 of Assignment 8.) There is a different method for representing \(A^k\) in terms of the eigenvectors and eigenvalues of \(A\). It is presented in section 4.8, eq. (6) and is illustrated in Example 6 of that section. Carry out the textbook’s procedure for this \(A\) with \(x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and hence show that

\[ x(k) = \left[ \left( (1 + \sqrt{5})/2 \right)^k - \left( (1 - \sqrt{5})/2 \right)^k \right] / \sqrt{5}. \]