Invisibility Cloaks via Non-Smooth Transformation Optics and Ray Tracing

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Abstract
We present examples of theoretically-predicted invisibility cloaks with shapes other than spheres and cylinders, including cones and ellipsoids, as well as shapes spliced from parts of these simpler shapes. In addition, we present an example explicitly displaying the non-uniqueness of invisibility cloaks of the same shape. We depict rays propagating through these example cloaks using ray tracing for geometric optics.

1. Introduction
In the past decade, the development of metamaterials, artificial composites of dielectrics with nano-size metallic inclusions, has made possible significant advances in transformation optics [1–7]. One of them was the design of an “invisibility cloak,” a metamaterial layer designed to guide plane, monochromatic, electromagnetic waves around a cavity in a perfect fashion so that they will emerge on the outgoing surface again as plane waves [8–18]. The first to be designed was a cylindrical cloak operating in the microwave range [13]. For optical wavelengths, a micron-size invisibility cloak was designed [19]. Theoretically, cloaking was predicted via ray optics using transformation properties of Maxwell’s equations [9, 20] and conformal transformations on Riemann surfaces [10, 11]. Even before, cloaking was predicted mathematically in [21] as nonuniqueness of conductivities that produce the same Dirichlet-to-Neumann map in a conducting medium. In most theoretical investigations, cloaks of highly symmetric shapes, such as spheres, cylinders, and squares, were considered [9–11, 14, 20].

Invisibility cloaking is achievable using metamaterials with appropriately designed, anisotropic dielectric permittivity and magnetic permeability tensors, which become singular at the inner boundary of the cloaking medium [13, 18, 20]. For each cloak shape, these tensors can be computed using a spatial transformation of the entire solid-cloak volume, less a point or a curve, into the hollow cloak shape. In general, such a transformation only has to be continuous, and has to equal the identity on the external surface of the cloak. For typical examples with simple shapes, such as spheres or cylinders, this transformation is smooth, even analytical [9, 20]. It is not unique, and therefore the dielectric permittivity and magnetic permeability tensors leading to the invisibility cloaking of a specific shape are also non-unique [18].

In this paper, we present new explicit examples of potential invisibility-cloak shapes that are splicings of simpler component shapes, such as hollow cylinders, and spherical and conical caps. The spatial trans-
formations leading to these cloaks are continuously matched but not smooth along the boundaries of the component shapes, and the corresponding dielectric permittivity and magnetic permeability tensors have discontinuities there. Prior to these, we also find the spatial transformations leading to conical and ellipsoidal shapes, and an alternative pair of dielectric permittivity and magnetic permeability tensors to those presented in [9, 20] for the spherical cloak. We demonstrate the cloaking properties of all these shapes using geometric ray optics for anisotropic media.

The remainder of the paper is organized as follows. In Section 2.1, we review the transformation properties of Maxwell’s equations that make it possible to compute the permittivity and permeability tensors for invisibility cloaks of arbitrary shapes. In Section 2.2, we review Hamiltonian ray optics for anisotropic media used to visualize light traveling through the cloaks. In Section 2.3, we derive Snell’s laws of refraction at the cloak surface. In Section 2.4, we review the results of [9, 20] on the spherical and cylindrical cloaks. In Section 3.1, we present an example of a spherical cloak with alternative permittivity and permeability tensors. In Sections 3.2 and 3.3, we describe conical and ellipsoidal cloaks. Finally, in Section 3.4, we present two examples of composite-shape cloaks with discontinuous permittivity and permeability tensors.

2. Background

In this section, we review the general theory of transformation optics that leads to invisibility cloaking. We discuss how the concept of cloaking can be reduced to a spatial transformation and then reinterpreted as a transformation of the dielectric permittivity and magnetic permeability of the cloaking medium. We then review Hamiltonian ray optics for anisotropic media, including the implications of Snell’s law of refraction at the interface. Finally, we review the results for the previously-described spherical and cylindrical cloaks, both for illustrative purposes and, more importantly, so that we can use them to assemble invisibility cloaks with more complicated shapes.

2.1. Transformation Optics Leading to Cloaking

Following [22, 23], we exploit the invariance of Maxwell’s equations under spatial coordinate transformations, as explained in the next few paragraphs. In particular, let us consider Maxwell’s equations in a medium with no sources or currents,

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \\
\nabla \cdot \mathbf{D} &= 0, & \nabla \cdot \mathbf{B} &= 0,
\end{align*}
\]

where \(\mathbf{E}\) represents the electric field, \(\mathbf{B}\) the magnetic induction, \(\mathbf{H}\) the magnetic field, \(\mathbf{D}\) the electric displacement field, and the constant \(c\) the speed of light in vacuum. The constitutive relations for a linear medium are given by the equations

\[
\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H},
\]

where \(\epsilon\) and \(\mu\) are the dielectric permittivity and magnetic permeability tensors, respectively.

If we change coordinates from \(\mathbf{x} = (x^1, x^2, x^3)\) to \(\mathbf{x}' = (x'^1, x'^2, x'^3)\), neither the form of Maxwell’s equations (1) nor the form of the constitutive relations (2) changes. What does change are the elements of the dielectric permittivity and magnetic permeability tensors, \(\epsilon\) and \(\mu\). In particular, if

\[
J_i^i = \frac{\partial x'^i}{\partial x^i}
\]

are the components of the Jacobian matrix \(J\) of the coordinate transformation, let us transform the electric and magnetic fields, \(\mathbf{E}\) and \(\mathbf{H}\), respectively, as

\[
E'^i = \sum_{i=1}^{3} J_i^i E^i, \quad H'^i = \sum_{i=1}^{3} J_i^i H^i,
\]
which, in vector-matrix notation, is $E' = J E, H' = J H$. If we let $A$ stand for either of two tensors $\epsilon$ and $\mu$, we likewise transform

$$A''^{ij} = (\det J)^{-1} \sum_{i,j=1}^{3} J_i^{i'} J_j^{j'} A^{ij}, \quad (5)$$

In matrix notation, equation (5) can be shown to read

$$A' = (\det J)^{-1} J A J^T, \quad (6)$$

where $T$ denotes the transpose, due to the fact that the tensor $A$ is symmetric. Under the transformations (4) and (5), Maxwell’s equations (1) retain their original form in the new coordinates [22, 23].

For an invisibility cloak, consider now a hollow region, say $\mathcal{R}'$, bounded on the inside and the outside by two convex surfaces and filled with an appropriate type of metamaterial. The cloak will be located in $\mathcal{R}'$, while the space inside the hole contained in $\mathcal{R}'$ will be invisible to observers outside of $\mathcal{R}'$. We assume the region outside of $\mathcal{R}'$ to be filled with air, which makes its dielectric permittivity and magnetic permeability scalars rather than tensors. Denote by $\mathcal{R}$ the region consisting of $\mathcal{R}'$ as well as the hole inside it, with the exception of a single point or curve inside this hole. Let us also denote by $\mathbf{x}'(x)$ the transformation of the region $\mathcal{R}$ into the region $\mathcal{R}'$, which we assume to be the identity at the outer boundary of $\mathcal{R}'$ and continuous on $\mathcal{R}$ except at the exceptional points. Its inverse $\mathbf{x}(x')$ clearly gives a parametrization of the region $\mathcal{R}$ (minus the singular point or curve) by a set of curvilinear coordinates in $\mathcal{R}'$. In particular, since we will be considering ray optics in this paper, we note that straight-line rays in $\mathcal{R}$, corresponding to plane waves, will be parametrized by curved rays in $\mathcal{R}'$. All the parametrizing curved rays will avoid the hole in $\mathcal{R}'$. (Rays passing through the exceptional point or curve in $\mathcal{R}$ cannot be described in these curvilinear coordinates, but they can safely be ignored.)

Using the above transformation rules, we can reinterpret the parametrizing curved rays in $\mathcal{R}'$ as physical rays propagating through a medium with the transformed electric permeability and magnetic permittivity tensors, $\epsilon'$ and $\mu'$, given by equation (5), with the corresponding $\epsilon$ and $\mu$ being scalars. This medium is in $\mathcal{R}'$, and no rays can enter the hole inside $\mathcal{R}'$ by construction. On the external boundary of $\mathcal{R}'$, the rays merge continuously with the straight-line rays in the surrounding air. In other words, the dielectric permittivity $\epsilon'$ and magnetic permeability $\mu'$ obtained from their isotropic counterparts using the mapping $\mathbf{x}(x')$ give correct optical characteristics of a metamaterial in the region $\mathcal{R}'$ so that this region becomes an invisibility cloak for the objects contained in the hole inside it [9, 20].

Note that the transformation $\mathbf{x}(x')$ leading to the invisibility cloak only needs to be piecewise smooth. In this case, the the electric permittivity $\epsilon'$ and magnetic permeability $\mu'$ in the cloaking medium have discontinuities, typically along two-dimensional surfaces.

2.2. Hamiltonian Ray Tracing in Anisotropic Media

In what is to follow, we only consider cloaking for single-frequency light. We assume its wavelength to be very short compared with the scales of the changes of the electric permittivity and magnetic permeability in the cloaking medium. In other words, we assume for the electric and magnetic fields the form

$$E = \mathcal{E} e^{i(\kappa S - \omega t)}, \quad H = \mathcal{H} e^{i(\kappa S - \omega t)}, \quad (7)$$

where $\kappa \gg 1$ is the wavenumber, $\omega = \kappa c$ is the frequency, $\mathcal{E}$ and $\mathcal{H}$ the complex, vector-valued amplitudes, and $S$ the position-dependent phase. Using equations (2), we thus transform Maxwell’s equations (1) into the equations [24]

$$\nabla \times (\mathcal{E} e^{i\kappa S}) = i \kappa \mu \mathcal{H} e^{i\kappa S}, \quad \nabla \times (\mathcal{H} e^{i\kappa S}) = -i \kappa \epsilon \mathcal{E} e^{i\kappa S}, \quad (8a)$$

$$\nabla \cdot (\epsilon \mathcal{E} e^{i\kappa S}) = 0, \quad \nabla \cdot (\mu \mathcal{H} e^{i\kappa S}) = 0. \quad (8b)$$

Expanding the curl and div terms in equations (8), canceling the exponentials, and keeping only the dominant $O(\kappa)$ terms (because $\kappa \gg 1$) yields the equations

$$\nabla S \times \mathcal{E} = \mu \mathcal{H}, \quad \nabla S \times \mathcal{H} = -\epsilon \mathcal{E}, \quad (9a)$$
\[ \epsilon \mathbf{E} \cdot \nabla \mathbf{S} = 0, \quad \mu \mathbf{H} \cdot \nabla \mathbf{S} = 0. \]  

(9b)

It should be clear that equations (9b) follow from equations (9a). From (9a) we finally derive the equations for \( \mathbf{E} \) and \( \mathbf{H} \) alone,

\[ \nabla \mathbf{S} \times \mu^{-1} (\nabla \mathbf{S} \times \mathbf{E}) = -\epsilon \mathbf{E}, \]  

(10a)

\[ \nabla \mathbf{S} \times \epsilon^{-1} (\nabla \mathbf{S} \times \mathbf{H}) = \mu \mathbf{H}, \]  

(10b)

To solve equations (10), we first define \( \Sigma \) to be the matrix of the operator \( \nabla \mathbf{S} \times \), and assume that both \( \epsilon \) and \( \mu \) have the identical form, say, \( \epsilon = \mu = A \). Multiplying them by \( A^{-1} \), we can then rewrite equations (10) in the form

\[ \left( (A^{-1} \Sigma)^2 + I \right) \mathbf{E} = \left( (A^{-1} \Sigma)^2 + I \right) \mathbf{H} = 0, \]  

(11)

for which the solvability condition is given by the equation

\[ \det \left( (A^{-1} \Sigma)^2 + I \right) = 0. \]  

(12)

Note that, in fact, since the amplitudes \( \mathbf{E} \) and \( \mathbf{H} \) must be linearly independent, equation (11) implies that \( 0 \) must be a double root of equation (12).

To compute the determinant in equation (12), it should be enough to compute the eigenvalues of the matrix \( (A^{-1} \Sigma)^2 \), and thus of the matrix \( A^{-1} \Sigma \). Computing the roots of the equation \( \det \left( A^{-1} \Sigma - \lambda I \right) = 0 \), where \( I \) is the identity matrix, is equivalent to computing the roots of the equation

\[ \det (\Sigma - \lambda A) = 0. \]  

(13)

We can compute the roots of this equation in the coordinates in which the tensor \( A \) is diagonal, which can be arrived at from any coordinates via an orthogonal transformation. Equation (13) then becomes

\[ \det \left[ \begin{array}{ccc} -\lambda a_1 & -S_z & S_y \\ S_z & -\lambda a_2 & -S_x \\ -S_y & S_x & -\lambda a_3 \end{array} \right] = 0, \]  

(14)

where the subscript denotes differentiation with respect to the indicated spatial variable.

The determinant in equation (14) can easily be evaluated without much calculation by using the properties of the respective matrix. In particular, the constant term in the resulting cubic polynomial equals \( \det \Sigma = 0 \), and the quadratic term also vanishes because there are no constant terms along the diagonal of the matrix in (14). The cubic term is clearly \( -\lambda^3 a_1 a_2 a_3 = -\lambda^3 \det A \). The linear term is the sum of the products of the terms \( -\lambda a_j \) with those terms in their corresponding minors which do not contain any more factors of \( \lambda \). Thus, this term is \( -\lambda \left( a_1 S_z^2 + a_2 S_y^2 + a_3 S_x^2 \right) \). Equation (14) thus becomes

\[ -\lambda \left( \lambda^2 \det A + \nabla \mathbf{S} \cdot A \nabla \mathbf{S} \right) = 0, \]  

(15)

and holds in any coordinate system.

From equation (15), we easily compute the eigenvalues of the matrix \( (A^{-1} \Sigma)^2 + I \) to be

\[ \lambda = 1, \quad \lambda = 1 - \frac{\nabla \mathbf{S} \cdot A \nabla \mathbf{S}}{\det A}, \]  

(16)

the second of which is double. Equation (12) therefore becomes

\[ \left( \frac{\nabla \mathbf{S} \cdot A \nabla \mathbf{S} - \det A}{\det A} \right)^2 = 0, \]

which finally gives the Hamilton-Jacobi equation

\[ \nabla \mathbf{S} \cdot A(x) \nabla \mathbf{S} - \det A(x) = 0 \]  

(17)
for the gradient $\nabla S$ of the phase $S$. Note that this equation can be multiplied by an arbitrary scalar function, $f(x)$, of the position vector $x$.

Equation (17) can be solved using characteristics, the light rays, as follows [25]. Denote $\nabla S = k$, and let us parametrize the characteristics by the parameter $\tau$. We then have

$$\frac{dk_i}{d\tau} = \sum_{j=1}^{3} \frac{\partial^2 S}{\partial x_i \partial x_j} \frac{dx_j}{d\tau}. \tag{18}$$

Let us write equation (17) in the abstract form $H(x, \nabla S) = H(x, k) = 0$, where

$$H(x, k) = f(x) [k \cdot A(x)k - \det A(x)], \tag{19}$$

and $f(x)$ is an arbitrary nonvanishing function of $x$. If we now differentiate the equation $H(x, \nabla S) = 0$ with respect to $x_i$, we find

$$\sum_{j=1}^{3} \frac{\partial H}{\partial k_j} \frac{\partial^2 S}{\partial x_i \partial x_j} + \frac{\partial H}{\partial x_i} = 0. \tag{20}$$

If we choose

$$\frac{dx_i}{d\tau} = \frac{\partial H}{\partial k_i},$$

equations (18) and (20) imply

$$\frac{dk_i}{d\tau} = -\frac{\partial H}{\partial x_i},$$

as well as

$$\frac{dH}{d\tau} = 0. \tag{21}$$

In other words, the rays corresponding to equation (17) are those trajectories of the Hamiltonian system

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial k}, \quad \frac{dk}{d\tau} = -\frac{\partial H}{\partial x}, \tag{22}$$

with the Hamiltonian (19), which lie on the surface $H(x, k) = 0$.

Observe that on the surface $H(x, k) = 0$, changing the function $f(x)$ in (19) only changes the parameter so that $d\tau \rightarrow f(x(\tau))d\tau$ and not the solutions of the ray equations (22). This corresponds to the fact that the original Hamilton-Jacobi equation (17) remains valid if it is multiplied by $f(x)$. Due to equation (21), we only need to ensure that the initial point $(x_0, k_0)$ on every trajectory satisfies the equation $H(x_0, k_0) = 0$. The phase $S$ can be recovered by integrating the expression

$$\frac{dS}{d\tau} = \sum_{i=1}^{3} \frac{\partial S}{\partial x_i} \frac{dx_i}{d\tau} = \sum_{i=1}^{3} k_i \frac{dx_i}{d\tau}$$

along all the trajectories. Note that $S$ does not have to be a single-valued function of $x$; its multi-valuedness can be absorbed in the amplitudes $E$ and $H$.

In the free space, we can take $f(x) = 1$ with no loss of generality, resulting in the Hamiltonian

$$H(x, k) = k^2 - 1, \tag{23}$$

and the corresponding Hamilton’s equations

$$\frac{dx}{d\tau} = k, \quad \frac{dk}{d\tau} = 0, \tag{24}$$

which result in straight rays.
2.3. Snell’s Law at the Air-Cloak Interface

At an interface where the transition between the two media is not smooth, a standard derivation identical to the one given on p. 125 of [24] shows that the tangential component of the vector $\mathbf{k}$ must remain preserved. In other words, if $\mathbf{n}$ is the unit normal to the interface at the point $\mathbf{x}$, then at this point,

$$\mathbf{n} \times (\mathbf{k}_1 - \mathbf{k}_2) = 0,$$

(25)

where $\mathbf{k}_1$ is the incident wave vector and $\mathbf{k}_2$ the refracted or reflected wave vector. This fixes two components of the vector $\mathbf{k}_2$. To find the third component, we note that the value of the Hamiltonian must be preserved across the interface,

$$H(\mathbf{x}, \mathbf{k}_1) = H(\mathbf{x}, \mathbf{k}_2) = 0.$$

(26)

Equation (26) is quadratic in the components of the wave vectors, so it gives two solutions: one for the refracted and one for the reflected wave, as depicted in Figure 1.

Figure 1: Refraction at the cloak surface: the wave vector $\mathbf{k}_1$ denotes the incident ray, $\mathbf{k}_{2(1)}$ denotes the reflected ray, and $\mathbf{k}_{2(2)}$ denotes the refracted ray. $H_1$ and $H_2$ are the Hamiltonians on their respective sides of the surface.

We use equations (25) and (26) to compute the change in the wave vector $\mathbf{k}$ both at the entry into and exit from the cloak. At an entry point $\mathbf{x}$ into the cloak, the vector $\mathbf{k}_1$ in the free space is known, while the vector $\mathbf{k}_2$ inside the cloak is unknown. Equation (25) implies that the difference of these two vectors must be proportional to the unit normal to the cloak:

$$\mathbf{k}_2 = \mathbf{k}_1 + \lambda \mathbf{n},$$

(27)

where $\lambda$ is a scalar parameter. After we have canceled the nonzero number $f(\mathbf{x})$ from the equation $H(\mathbf{x}, \mathbf{k}_2) = 0$, with $H(\mathbf{x}, \mathbf{k})$ as in equation (19), we find that the vector $\mathbf{k}_2$ satisfies the equation

$$\mathbf{k}_2 \cdot A(\mathbf{x})\mathbf{k}_2 - \det A(\mathbf{x}) = 0.$$

Together with equation (27), this equation gives a quadratic equation for the parameter $\lambda$, which is

$$\lambda^2 \mathbf{n} \cdot A(\mathbf{x})\mathbf{n} + 2\lambda \mathbf{k}_1 \cdot A(\mathbf{x})\mathbf{n} + \mathbf{k}_1 \cdot A(\mathbf{x})\mathbf{k}_1 - \det A(\mathbf{x}) = 0.$$
and yields,
\[
\lambda = -\frac{k_1 \cdot A(x)n}{n \cdot A(x)n} \pm \sqrt{\left(\frac{k_1 \cdot A(x)n}{n \cdot A(x)n}\right)^2 + \frac{\det A(x) - k_1 \cdot A(x)k_1}{n \cdot A(x)n}}. \tag{28}
\]

The choice of the sign must be such that the vector \(k_2\) points into the cloaking medium.

At an exit point \(x\) from the cloaking medium, the wave vector \(k_1\) inside the medium is known, and the unknown free-space wave vector \(k_2\) satisfies the relation (27) as well as the equation (23), \(k_2^2 - 1 = 0\). These equations imply the quadratic equation
\[
\lambda^2 + 2\lambda k_1 \cdot n + k_1^2 - 1 = 0,
\]
in which we have taken into account that \(|n| = 0\). The values of the parameter \(\lambda\) obtained from this equation are
\[
\lambda = -k_1 \cdot n \pm \sqrt{(k_1 \cdot n)^2 + 1 - k_1^2}, \tag{29}
\]
where the sign is now taken so that the wave vector \(k_2\) points out of the cloaking medium.

2.4. Review of the Results for the Spherical and Cylindrical Cloaks

The general methods for obtaining the dielectric permittivity and magnetic permeability tensors, \(\epsilon\) and \(\mu\), developed in the preceding sections for general invisibility cloak geometry, can be illustrated in the simplest fashion using the spherical and cylindrical cloaks. In addition, we will use the results for these simple geometries in the later sections as building blocks, as well as to guide our derivations in more complicated geometries. Therefore, we here review these known results, mainly following the treatment in [9, 20].

2.4.1. Spherical Cloak

For the spherical cloak, we choose the shell \(a < r' < b\), and transform the punctured sphere \(0 < r < b\) into this shell via the transformation
\[
r' = \frac{b - a}{b} r + a. \tag{30}
\]
Here, \(r = |x|\) the radial variable in original space and \(r' = |x'|\) the radial variable in the transformed space. This transformation leaves the angle variables intact, which is equivalent to leaving intact unit vectors in the radial direction,
\[
\frac{x'}{r'} = \frac{x}{r}. \tag{31}
\]
From equations (30) and (31), we find the explicit form of the transformation as
\[
x' = \left(\frac{b-a}{b} + \frac{a}{r}\right) x, \quad 0 < r < b. \tag{32}
\]
Since we use the identity transformation outside the punctured sphere \(0 < r < b\), we note that the transformation (30) is continuous on the boundary \(r = b\), but not smooth.

Taking the derivatives of the transformation (32), we find its Jacobian matrix \(J\) to be
\[
J = \left(\frac{b-a}{b} + \frac{a}{r}\right) I - \frac{a}{r^2} x \otimes x,
\]
where \(x \otimes y = xy^T\) denotes the tensor product. Since we will eventually need the expression for \(J\) in terms of the transformed coordinates \(x'\), we use equations (30) and (31) to find
\[
J = \frac{1}{r} \left(r'I - \frac{a}{r^2} x' \otimes x'\right). \tag{33}
\]
To find the determinant of the Jacobian matrix $J$, we take into account the spherical symmetry of the problem and compute it at the point $\mathbf{x}' = (r', 0, 0)^T$, where $J = \text{diag}(r' - a, r', r')/r$, where $\text{diag}(\cdot)$ denotes a diagonal matrix. This calculation yields

$$\det J = \frac{(r' - a)r'^2}{r^3} = \frac{(b - a)b}{r^2} \left( \frac{r'}{r'} \right)^2 = \left( \frac{b}{b} \right)^3 \left( \frac{r'}{r'} - a \right)^2. \quad (34)$$

To find the dielectric permittivity and magnetic permeability tensors, we use equation (5) and the fact that in the $x$-coordinates both these tensors are equal to the identity, $\epsilon = \mu = A = I$. Equation (5) implies that

$$A' = (\det J)^{-1} J J^T = (\det J)^{-1} J^2, \quad (35)$$

with the last equality holding because $J$ is symmetric. Using equations (33) and (34), and the fact that $(\mathbf{x}' \otimes \mathbf{x}')^2 = r'^2 \mathbf{x}' \otimes \mathbf{x}'$, as well as dropping the primes on the transformed variables for convenience of notation, we arrive at the dielectric permittivity and magnetic permeability tensors

$$\epsilon = \mu = \frac{b}{b - a} \left( 1 - \frac{2ar - a^2}{r^4} \mathbf{x} \otimes \mathbf{x} \right). \quad (36)$$

Using equation (35) and the rule for calculating determinants of matrix products, we find

$$\det \epsilon = (\det J)^{-1}. \quad (37)$$

Using equation (19) with $A(\mathbf{x}) = \epsilon = \mu$ from equation (36) and $\det A(\mathbf{x})$ from equation (37), and choosing

$$f(\mathbf{x}) = \frac{1}{2} \frac{b - a}{b},$$

we find the Hamiltonian to be

$$H = \frac{1}{2} k^2 - \frac{1}{2} \frac{2ar - a^2}{r^4} (\mathbf{x} \cdot \mathbf{k})^2 - \frac{1}{2} \left[ \frac{b(r - a)}{r(b - a)} \right]^2. \quad (38a)$$

The derivatives involved in Hamilton’s equations of motion now follow as

$$\frac{\partial H}{\partial k} = k - \frac{2ar - a^2}{r^4} (\mathbf{x} \cdot \mathbf{k}) \mathbf{x}, \quad (38a)$$

$$\frac{\partial H}{\partial \mathbf{x}} = -\frac{2ar - a^2}{r^4} (\mathbf{x} \cdot \mathbf{k}) k + \frac{3ar - 2a^2}{r^5} (\mathbf{x} \cdot \mathbf{k})^2 \mathbf{x} - \left( \frac{b}{b - a} \right)^2 \left( \frac{ar - a^2}{r^4} \right) \mathbf{x}. \quad (38b)$$

The corresponding set of ordinary differential equations, (22), is solved numerically with a standard fourth-order Runge-Kutta method, implemented in Matlab. On the outer boundary $r = b$ of the spherical cloak we use formulas (27), (28), and (29) to refract the rays. The resulting rays are displayed in Figure 2, which shows how they curve around the inner sphere, and continue collinearly with the corresponding incoming rays after they have exited the outer spherical surface of the cloak. Our results agree with those of [9, 20].

2.4.2. Cylindrical Cloak

For an infinite cylindrical cloak with the rotational symmetry axis aligned along the $z$-axis, the coordinate transformation is given by the two-dimensional analog of equations (30), (31), and (32). In particular, if $\rho = (x, y, 0)$ and $\rho' = (x', y', 0)$, we put

$$\rho' = \frac{b - a}{b} \rho + a$$

and

$$\frac{\rho'}{\rho} = \frac{2}{\rho}. \quad (39)$$
so that
\[
\rho' = \left( \frac{b-a}{b} + \frac{a}{b} \right) \rho, \quad 0 < \rho < b.
\] (40)

The z-coordinate is unchanged: \( z' = z \). The Jacobian matrix \( J \) is computed in the same way as in the previous section, and is given by the expression
\[
J = \frac{1}{\rho} \left( \rho' P - \frac{a}{\rho^2} \rho' \otimes \rho' \right) + Z,
\] (41)
where
\[
P = \text{diag}(1, 1, 0), \quad Z = \text{diag}(0, 0, 1);
\] (42)
its determinant is computed at the point \( \rho' = (\rho', 0, 0) \) and equals
\[
\det J = \frac{(\rho' - a)\rho'}{\rho^2} = \frac{b - a}{b} \rho' = \left( \frac{b-a}{b} \right)^2 \rho' - b.
\] (43)

Using equations (35), (41), and (43), and dropping the primes on the transformed variables, we calculate the dielectric permittivity
\[
\epsilon = \frac{\rho}{\rho-a} P - \frac{2a\rho-a^2}{\rho^2(\rho-a)} \rho \otimes \rho + \left( \frac{b}{b-a} \right)^2 \left( \frac{b-a}{\rho} \right) Z,
\]
with the determinant \( \det \epsilon = (\det J)^{-1} \). Letting
\[
f(x) = \frac{1}{2} \frac{\rho-a}{\rho},
\]
and using \( A(x) = \epsilon = \mu \) in equation (19), we thus find the resulting Hamiltonian
\[
H = \frac{1}{2} \mathbf{k} \cdot P \mathbf{k} - \frac{1}{2} \frac{2a\rho-a^2}{\rho^2(\rho-a)} (\mathbf{\rho} \cdot \mathbf{k})^2 + \frac{1}{2} \left[ \frac{b(b-a)}{\rho(b-a)} \right]^2 (\mathbf{k} \cdot Z \mathbf{k} - 1),
\]
and its partial derivatives
\[
\frac{\partial H}{\partial \mathbf{k}} = P \mathbf{k} - \frac{2a\rho-a^2}{\rho^2(\rho-a)} (\mathbf{\rho} \cdot \mathbf{k}) \mathbf{\rho} + \left[ \frac{b(b-a)}{\rho(b-a)} \right]^2 Z \mathbf{k},
\] (44a)
\[
\frac{\partial H}{\partial x} = \frac{3a\rho - 2a^2}{\rho^6}(\rho \cdot k)^2 \rho - \frac{2a\rho - a^2}{\rho^4}(\rho \cdot k)Pk + \left(\frac{b}{b-a}\right)^2 \frac{(a\rho - a^2)}{\rho^4} (k \cdot Zk - 1)\rho.
\]

Again, we solve the corresponding set of Hamilton’s ordinary differential equations, (22), numerically using Matlab, and use equations (27), (28), and (29) on the outer boundary \( \rho = b \) of the cylindrical cloak to refract the rays. The resulting rays are shown in Figure 2. Note that the z-slope of the rays decreases as they approach the inner boundary of the cloak, and increases near its outer boundary. Our results agree with those of [20].

3. Results

We now present illustrative examples of new possible geometries and coordinate transformations that would give rise to invisibility cloaks, which are the main results of this paper. The first among them is an alternative coordinate transformation, and thus an alternative permittivity and permeability tensor, leading to a spherical cloak, which illustrates the non-uniqueness of the coordinate transformations leading to invisibility cloaking. The second is a derivation of conical and ellipsoidal cloaks, which show that geometries other than spherical and cylindrical can still give rise to invisibility cloaking. And finally, we display two invisibility cloaks for which the coordinate transformations leading to them are not smooth, and thus their dielectric permittivity and magnetic permeability profiles are discontinuous.

3.1. Alternative Permittivity and Permeability Tensors for the Spherical Cloak

To demonstrate non-uniqueness of cloaking for objects of a given shape, we here demonstrate an alternative coordinate transformation leading to cloaking for the case of the sphere. Recall that the transformation (30) used to derive the ray equations for the spherical cloak, which we described in the previous section, was radial and linear.

Here, we still use a radial transformation to create a spherical invisibility cloak, but employ the alternative, quadratic, radial scaling

\[
r' = \frac{b-a}{b^2} r^2 + a
\]

instead of (30). Unit vectors in the radial direction remain intact, so that equation (31) still holds, and so the transformation from the \( x \)- to the \( x' \)-variables becomes

\[
x' = \left(\frac{b-a}{b^2} r + \frac{a}{r}\right) x, \quad 0 < r < b.
\]

Using (46), and then also (31) and (45), we find the Jacobian matrix

\[
J = \frac{r'}{r} I + \left(\frac{b-a}{b^2} r - \frac{a}{r}\right) \frac{x \otimes x}{r^2} = \frac{1}{r} \left( r' I + (r' - 2a) \frac{x' \otimes x'}{r'^2} \right).
\]

By symmetry, we can again compute its determinant at the special point \( x' = (r', 0, 0) \), and find

\[
det J = \frac{2(r' - a)r'^2}{r^3} = \frac{2r'^2}{b^3} \sqrt{\frac{(b-a)^3}{r'^2 - a}}.
\]

From equation (35), and dropping the primes on the transformed variables, we then find the permittivity and permeability \( \epsilon = \mu = A(x) \) to be

\[
\epsilon = \frac{b}{2 \sqrt{(r-a)(b-a)}} \left[ 1 + \frac{3r'^2 - 8ar + 4a^2}{r^4} \frac{x \otimes x}{r'^2} \right],
\]

with the determinant \( det \epsilon = (det J)^{-1} \).
The Hamiltonian (19) can now be computed as

\[
H = \frac{1}{2} k^2 + \frac{1}{2} \frac{3r^2 - 8ar + 4a^2}{r^4} (k \cdot x)^2 - \frac{1}{2} \frac{b^2}{r^2} (r - a)
\]

where the scaling function \( f(x) \) is chosen to be

\[
f(x) = \frac{\sqrt{(r - a)(b - a)}}{b}.
\]

The partial derivatives needed in Hamilton’s equations are readily calculated to be

\[
\frac{\partial H}{\partial k} = k + \frac{3r^2 - 8ar + 4a^2}{r^4} (k \cdot x)x,
\]

\[
\frac{\partial H}{\partial x} = \frac{3r^2 - 8ar + 4a^2}{r^4} (x \cdot k)k - \frac{3r^2 - 12ar + 8a^2}{r^6} (x \cdot k)^2 x - \frac{1}{2} \frac{b^2}{b - a} \frac{2a - r}{r^4} x.
\]

The resulting rays are depicted in Figure 3. From this figure, it is evident that the rays in the case of the quadratic scaling (45) are bent less than those in the case of the linear scaling (30). The rays near the polar ray, which is not deflected around the inner sphere, are therefore more compressed in the case of the quadratic scaling.

**3.2. Conical Cloak**

We next find the dielectric permeability and magnetic permittivity and compute light rays propagating through a conical cloak. The cone is a shape with a rotational and scaling, rather than a spherical or rotational and translational, symmetry, which is reflected in the coordinate transformation generating a conical invisibility cloak.

If we position the cone so that its axis is the \( z \)-axis, we see that one coordinate transformation that would give rise to a conical invisibility cloak leaves the \( z \)-coordinate alone, and maps each punctured circle \( 0 < \rho < b \) onto the corresponding annulus \( az < \rho' < bz \), where again \( \rho = (x, y, 0) \) and \( \rho' = (x', y', 0) \). The transformation

\[
\rho' = \left( \frac{b - a}{b} \rho + az \right)
\]

accomplishes the radial change. Again, as for the cylindrical cloak, equation (39) must hold for unit vectors in the direction perpendicular to the \( z \)-axis, so that the entire transformation of the \( x \)- and \( y \)-coordinates can be written as

\[
\rho' = \left( \frac{b - a}{b} + \frac{az}{\rho} \right) \rho, \quad 0 < \rho < b.
\]
Calculating the appropriate partial derivatives yields the Jacobian

\[ J = \frac{\rho'}{\rho} P + a \frac{\rho}{\rho} \otimes e_z - \frac{az}{\rho} \frac{\rho \otimes \rho}{\rho^2} + Z = \frac{1}{\rho} \left[ \rho' P - az' \frac{\rho' \otimes \rho'}{\rho'^2} \right] + a \frac{\rho'}{\rho} \otimes e_z + Z, \]

where \( e_z = (0, 0, 1) \) is the unit vector in the z-direction. Its determinant equals its 2 \times 2 sub-determinant in the upper left corner, since its last row equals the vector \( e_z \). This sub-determinant can easily be computed for \( \rho' = (\rho', 0, 0) \) to give

\[ \det J = \frac{\rho'(\rho' - az')}{\rho'^2} = \frac{\rho'}{\rho' - az'} \left( \frac{b-a}{b} \right)^2. \]

To compute the permittivity \( \epsilon \), we use equation (5) in the original form

\[ \epsilon' = (\det J)^{-1} J J^T \] (49)

because the Jacobian matrix \( J \) is not symmetric for the cone, and because the permittivity in the untransformed coordinates still equals unity. Dropping the primes on the transformed variables, we thus derive

\[ \epsilon = \frac{\rho}{\rho - az - 2a\rho z - a^2 z^2} P - \frac{a^2 \rho z - a^2 z^2}{\rho^4 (\rho - az)} \rho \otimes \rho \]

\[ + \left( \frac{b}{b - a} \right)^2 \left( \frac{\rho - az}{\rho} \right) \left[ \frac{\rho^2 (\rho \cdot k)^2}{\rho^4} + \frac{2a}{\rho} (\rho \cdot k)(e_z \cdot k) + k \cdot Zk - 1 \right], \]

with the determinant \( \det \epsilon = (\det J)^{-1} \), which is easily derived from the formula (49) and the rules for determinants of matrix products and transposes.

To calculate the Hamiltonian (19), we use the factor

\[ f(x) = \frac{\rho - az}{2\rho} \]

and \( A(x) = \epsilon = \mu \) from equation (49), to obtain

\[ H = \frac{1}{2} k \cdot P k - \frac{1}{2} \frac{2a\rho z - a^2 z^2}{\rho^4} (\rho \cdot k)^2 \]

\[ + \frac{1}{2} \left( \frac{b(\rho - az)}{\rho(b - a)} \right)^2 \left[ \frac{\rho^2 (\rho \cdot k)^2}{\rho^4} + \frac{2a}{\rho} (\rho \cdot k)(e_z \cdot k) + k \cdot Zk - 1 \right], \]

with the matrices \( P \) and \( Z \) again expressed as in equation (42). The partial derivatives entering Hamilton’s equations of motion are therefore given by the expressions

\[ \frac{\partial H}{\partial k} = P k - \frac{2az - a^2 z^2}{\rho^4} (\rho \cdot k) \rho + \left( \frac{b(\rho - az)}{\rho(b - a)} \right)^2 \left[ \frac{\rho^2 (\rho \cdot k)^2}{\rho^4} + \frac{2a}{\rho} (\rho \cdot k)(e_z \cdot k) + k \cdot Zk + Zk \right], \] (50a)

\[ \frac{\partial H}{\partial \rho} = \frac{3a \rho z - a^2 z^2}{\rho^4} (\rho \cdot k)^2 \rho - \frac{2a \rho z - a^2 z^2}{\rho^4} (\rho \cdot k) P k \]

\[ + \left( \frac{b}{b - a} \right)^2 \left( \frac{a \rho z - a^2 z^2}{\rho^4} \right) \left[ \frac{\rho^2 (\rho \cdot k)^2}{\rho^4} + \frac{2a}{\rho} (\rho \cdot k)(e_z \cdot k) + k \cdot Zk - 1 \right] \rho \]

\[ + \left( \frac{b(\rho - az)}{\rho(b - a)} \right)^2 \left[ - \frac{\rho^2 (\rho \cdot k)^2}{\rho^4} \rho + \frac{2a}{\rho^3} (\rho \cdot k) P k - \frac{a}{\rho} (\rho \cdot k)(e_z \cdot k) \rho + \frac{a}{\rho} (e_z \cdot k) P k \right], \] (50b)

\[ \frac{\partial H}{\partial z} = \frac{a^2 z - a \rho}{\rho^4} (\rho \cdot k)^2 \]

\[ + \left( \frac{b}{b - a} \right)^2 \left( \frac{a^2 z - a \rho}{\rho^2} \right) \left[ \frac{\rho^2 (\rho \cdot k)^2}{\rho^4} + \frac{2a}{\rho} (\rho \cdot k)(e_z \cdot k) + k \cdot Zk - 1 \right]. \] (50c)
3.3. Ellipsoidal Cloak

An invisibility cloak in the form of a triaxial ellipsoid only has three discrete symmetries, which are mirroring across the planes spanned by pairs of its principal axes, and no continuous symmetry.

As the cloak is assumed to lie in the ellipsoidal shell

$$\alpha^2 < \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < \beta^2,$$

it is simplest to first transform the problem to the spherical case by applying the linear transformation $\xi = \Omega x$, with $\Omega = \text{diag}(1/a, 1/b, 1/c)$. In this way, the ellipsoidal shell containing the cloaking medium is mapped onto the spherical shell $\alpha < \xi < \beta$, with $\xi = |\xi|$. The mapping of the punctured ellipsoidal solid

$$0 < \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < \beta^2$$

onto the above ellipsoidal shell is now accomplished via the mapping (30) of the punctured sphere $0 < \xi < \beta$ onto the spherical shell $\alpha < \xi < \beta$. It is immediately clear that the Jacobian matrix of the transformation generating the ellipsoidal cloak equals

$$J(x) = \Omega^{-1} J(\xi) \Omega,$$

where $J(\xi)$ is the Jacobian matrix corresponding to the spherical cloak, as described by equation (33).

Figure 4: Rays propagating through conical (left) and ellipsoidal invisibility cloaks (right).

We use equations (49) and (51), as well as the symmetry of the tensors $\Omega$ and $J(\xi)$, to find the expression for the permittivity tensor

$$\epsilon = [\det J(\xi)]^{-1} \Omega^{-1} J(\xi) \Omega^2 J(\xi) \Omega^{-1}.$$ 

In this way, we find the permittivity tensor to be

$$\epsilon = \frac{\beta}{\beta - \alpha} \left( I - \frac{\alpha}{|\Omega x|^3} (\mathbf{x} \otimes \Omega^2 \mathbf{x} + \Omega^2 \mathbf{x} \otimes \mathbf{x}) + \frac{\alpha^2}{|\Omega x|^6} |\Omega^2 \mathbf{x}|^2 \mathbf{x} \otimes \mathbf{x} \right),$$

and its determinant to equal

$$\det \epsilon = \det [J(\xi)]^{-1} = \left( \frac{\beta}{\beta - \alpha} \right)^3 \left( \frac{|\Omega x|}{|\Omega x|} - \alpha \right)^2.$$
Letting
\[ f(x) = \frac{1}{2} \frac{\beta - \alpha}{\beta} \]
and using \( A(x) = \epsilon = \mu \) from equation (52), we find the Hamiltonian (19) to be
\[
H = \frac{1}{2} k^2 - \frac{\alpha}{|\Omega x|^3} (x \cdot k)(\Omega^2 x \cdot k) + \frac{1}{2} \frac{\alpha^2 |\Omega^2 x|^2}{|\Omega x|^6} (x \cdot k)^2 - \frac{1}{2} \left[ \frac{\beta(|\Omega x| - \alpha)}{|\Omega x|} \right]^2.
\]
The partial derivatives of this Hamiltonian,
\[
\frac{\partial H}{\partial k} = k - \frac{\alpha}{|\Omega x|^3} [(x \cdot k)\Omega^2 x + (\Omega^2 x \cdot k)x] + \frac{\alpha^2 |\Omega^2 x|^2}{|\Omega x|^6} (x \cdot k)x,
\]
\[
\frac{\partial H}{\partial x} = -\frac{\alpha}{|\Omega x|^3} [(\Omega^2 x \cdot k)k + (x \cdot k)\Omega^2 k] + \frac{\alpha^2 |\Omega^2 x|^2}{|\Omega x|^6} (x \cdot k)k
\]
\[
+ \left[ \frac{3\alpha}{|\Omega x|^5} (x \cdot k)(\Omega^2 x \cdot k) - \frac{3\alpha^2 |\Omega^2 x|^2}{|\Omega x|^8} (x \cdot k)^2 - \alpha \left( \frac{\beta}{\beta - \alpha} \right)^2 \frac{|\Omega x| - \alpha}{|\Omega x|^4} \right] \Omega^2 x
\]
\[
+ \frac{\alpha^2}{|\Omega x|^6} (x \cdot k)^2 \Omega^4 x,
\]
give Hamilton’s equations of motion (22).

A set of parallel rays passing through the ellipsoidal invisibility cloak is shown in Figure 4. We have chosen these rays not to be parallel to any of the three principal axes of the ellipsoidal shell, yet they still exit the cloak collinearly with the incident ray. Thus, we find that the ellipsoidal shape gives an invisibility cloak without a continuous symmetry.

### 3.4. Invisibility Cloaking via Piecewise Smooth Transformations

As mentioned in Section 2, even though a transformation leading to an invisibility cloak should be continuous, it only needs to be piecewise smooth, resulting in dielectric permittivity and magnetic permeability tensors which are discontinuous, typically across a set of two-dimensional surfaces. Across these surfaces, we use Snell’s law to refract the rays again as we would when they enter through the exterior surface of the cloak, in particular, using equations (27) and (28).

Our first computation involves rays around a semi-infinite cylinder with a spherical cap. The transformation \( x'(x) \) is a piecewise splicing of the transformations (32) and (40), and maps the solid version of this object less the center half-ray of the cylinder to the hollow cloak. It is continuous but not smooth at the interface between the cylinder and the cap. To compute the rays, we use Hamilton’s equations obtained from the derivatives of the respective Hamiltonians, (38) and (44), presented at the ends of Sections 2.4.1 and 2.4.2, in the appropriate portions of the cloak. In Figure 5, we display rays that pass through both the cylinder and the cap and avoid the internal hollow cavity. Note that they are not smooth at the interface between the cylinder and the cap.

Likewise, we can splice together a solid cylinder with a spherical cap on one end and a conical cap on the other. We then expand the joint symmetry axis of the cylinder and cone into a hollow of the same shape, with the hollowed-out pencil-like shape serving as an invisibility cloak, as shown in Figure 5. For this transformation, we use a piecewise splicing of the mappings (32), (40), and (50). Again, the transformation is continuous but not smooth at the interfaces between the cylinder and the spherical and conical caps. Rays passing through this pencil-shaped cloak around the hollow cavity inside it are displayed in Figure 5, from which one can again see that they are not smooth as they pass through the internal interfaces.
4. Conclusions

We have presented theoretically-computed examples of invisibility cloaks in a number of shapes with varying degrees of symmetry. Some of these cloaks can be composed of parts along whose interfaces the dielectric permittivity and magnetic permeability tensors have discontinuities, which arise from the non-smoothness of the mapping that is used to devise the cloaking region. Nevertheless, our numerical results confirm that perfect cloaking for single-frequency light can still be attained for their shapes. Full wave simulations could be carried out for these new example cloak shapes along the lines of [14, 15].

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