From baking a cake to solving the diffusion equation

Edward A. Olszewski

Department of Physics, University of North Carolina at Wilmington, Wilmington, North Carolina 28403-5606

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We explain how modifying a cake recipe by changing either the dimensions of the cake or the amount of cake batter alters the baking time. We restrict our consideration to the génoise and obtain a semiempirical relation for the baking time as a function of oven temperature, initial temperature of the cake batter, and dimensions of the unbaked cake. The relation, which is based on the diffusion equation, has three parameters whose values are estimated from data obtained by baking cakes in cylindrical pans of various diameters. The relation takes into account the evaporation of moisture at the top surface of the cake, which is the dominant factor affecting the baking time of a cake. © 2006 American Association of Physics Teachers.

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I. INTRODUCTION

To cook and do it well requires little knowledge of the sciences in general and physics in particular. Indeed, most cookbooks are just that—cookbooks: one merely follows the steps in a recipe and obtains the finished product. Usually no discussion of the physical principles on which a recipe is based is given, making it difficult in many instances to modify the recipe in nontrivial ways. There are, however, noteworthy exceptions.1–4

Surprisingly, there appears to be a dearth of research related to the cake baking process. Much of what appears in the literature is specifically related to bread baking.5,6 A notable exception is the work of Lostie and collaborators.7,8 They analyzed the baking process of sponge batter during the first baking period.9,10 As described in Ref. 7 the data used in their analysis were obtained from controlled experiments in which the cake batter was heated only from the top, with the sides and bottom of the pan thermally insulated. Based on their experiments they proposed a one-dimensional model of the baking process to predict the spatial dependence and time evolution of water content (water and vapor), temperature, gas phase pressure, and porosity (the proportion of non-solid volume to total volume) of the cake batter. The model has nine adjustable parameters whose values are determined by fitting the model to their data.

The motivation for this article is to provide an answer to the question of how the baking time of a cake varies when its recipe is modified, either by changing the dimensions of the cake or the amount of cake batter. To make the problem manageable we have restricted our analysis to one type of cake, the génoise, one of the basic cakes of classic French cuisine. Although the technique for baking this cake is somewhat exacting, the ingredients are readily available, facilitating attempts to reproduce our results. The theoretical basis of our solution is the diffusion equation.

The outline of the article is as follows. First, we briefly discuss the quantitative aspects of cake baking. Next, we describe the experimental techniques and data used in the subsequent analysis. We, then, propose a simple model for the cake baking process and use it to obtain a semiempirical formula for estimating the baking time of the génoise.

II. CAKE BAKING AS A DIFFUSION PROCESS

A. Cake baking from a quantitative point of view

Before modeling the cake baking process, we first make more precise the imprecise measuring techniques that have customarily accompanied the instructions for baking a cake. A typical cake recipe lists the ingredients with amounts, a description of the technique used to prepare the cake batter, and the baking time for the recipe. Until recently, the amounts of the ingredients have generally been given in units of liquid measure, such as the cup or tablespoon. We can surmise that the reason for these units has been the scarcity of mass measuring scales in the home. The reason for using dry measure (measuring by scale) rather than liquid measure is to increase the likelihood of consistently producing a cake with the desired taste, texture, and appearance. Generally, the time necessary to bake a cake is accompanied by some subjective measure of determining whether the cake has baked to the proper degree of doneness. Such subjective measures include when a toothpick thrust into the center of the cake comes out clean, when the cake shows a faint line of shrinkage from the sides of the pan, when the top springs back lightly, and when you can smell the cake. To a first approximation these criteria can be quantified by equating the degree of doneness to the temperature at the center of the cake, although we can imagine more complex criteria that include other measurable quantities such as the temperature gradient.

B. The génoise

The recipe for the génoise is adapted from that described by Pépin.11–13 In Table I we converted the units to dry measure. Qualitatively, we have found that the recipe prepared using traditional measuring techniques, that is, proportions of ingredients are measured using measuring cups (liquid measure) rather than a scale (dry measure), produces a cake consistent in texture, taste, and appearance with one prepared using dry measuring techniques. Quantitatively, the amounts measured and re-measured using traditional techniques typically differ by less than 2% compared to amounts measured by weight.

According to the recipe the cake batter is placed in two cake pans, 8 in. in diameter by 1.5 in. deep, filling each pan 3/4 full. The cakes are baked in an oven of temperature $T_b = 350\, \text{°F}$ for a time between 22 and 25 min. To establish
Table I. The proportion of ingredients for the génoise is given in traditional measure as well as dry measure.

<table>
<thead>
<tr>
<th>Ingredient</th>
<th>Traditional measure</th>
<th>Dry measure (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eggs</td>
<td>6 large</td>
<td>298</td>
</tr>
<tr>
<td>Sugar</td>
<td>3/4 cup</td>
<td>176</td>
</tr>
<tr>
<td>Vanilla extract</td>
<td>1/2 tsp</td>
<td>2</td>
</tr>
<tr>
<td>All purpose flour</td>
<td>1 cup</td>
<td>144</td>
</tr>
<tr>
<td>Butter</td>
<td>3/4 stick</td>
<td>114</td>
</tr>
</tbody>
</table>

benchmarks for this study we have prepared the recipe, with exceptions as noted, filling two 8 in. cake pans to a depth of 1 in. We assessed the degree of doneness using traditional techniques and observed the baking time to be approximately 17 min. The qualitative measure of doneness corresponds to a temperature \( T_f = 203 \, ^\circ F \) at the center of the cake. The recipe was prepared several times with baking times varying by no more than 2 min.

C. Theory from a naive perspective

We initially believed that we could model the cake baking process as a simple diffusion process. Although this model is inadequate, it is the basis of the final model and is thus presented. This model is based on the diffusion equation,

\[
D \nabla^2 T = \frac{\partial T}{\partial t},
\]

where \( T = T(t, r, \theta, z) \) is the temperature of the cake at time \( t \) at the position \((r, \theta, z)\) (in cylindrical coordinates) within the cake, and \( D \) is the heat diffusivity (assumed constant) of the cake batter.

We assume the cake batter to be in a cylindrical pan of radius \( R \) and thickness \( Z \) at initial temperature \( T_i \) and baked in an oven of constant temperature, \( T_b \). We solve the diffusion equation with these initial conditions for the temperature \( T \), which is independent of \( \theta \) because of azimuthal symmetry,

\[
T(t, r, z) = T_b + \left(T_f - T_b\right) \frac{2R}{\pi} \sum_{n=1, m=1}^{\infty} \frac{1}{n} \Phi_{mn},
\]

where the \( \Phi_{mn} \) are the normalized eigenfunctions of the diffusion equation,

\[
\Phi_{mn} = \frac{2}{RJ_1(x_m,0)} \sin \left( \frac{n \pi x}{Z} \right) \frac{J_0(x_m,0)}{J_1(x_m,0)} \exp \left[ -\left( \frac{n \pi}{Z} \right)^2 + \frac{\left( \frac{x_m}{R} \right)^2}{2} \right] Df, \]

and \( x_m,0 \) is the \( m \)th root of the zeroth-order Bessel function. In Eq. (2) the first term on the right-hand side is the steady state solution. The second term is the solution particular to the boundary conditions and vanishes at the surface bounding the cylinder. Only the Bessel functions of zeroth order appear because of azimuthal symmetry. The other factors are required to reproduce the initial temperature in the interior of the cake at \( t=0 \).

The initial temperature of the cake batter \( T_i = 80 \, ^\circ F \) corresponds to the temperature at which the cake batter is prepared. The initial and final temperatures of the cake and temperature of the oven satisfy the relation

\[
\frac{T_f - T_i}{T_b - T_i} < 1.
\]

For temperatures \( T_f \) that satisfy Eq. (4), the infinite series in Eq. (2) evaluated at the center of the cake can be approximated by a single term

\[
T(t_f, 0, Z/2) = T_f = T_b + \left(T_f - T_b\right) F,
\]

where

\[
F = \begin{cases}
\frac{8}{\pi x_{1,0} f_{1}(x_{1,0})} \exp \left[ -\left( \frac{n \pi}{Z} \right)^2 + \left( \frac{x_{1,0}}{R} \right)^2 \right] Df, & \text{(C1 \leq 1 or C2 \leq 1)} \\
\frac{4}{\pi} \exp \left[ -\left( \frac{n \pi}{Z} \right)^2 Df \right], & \text{(C1 > 1)} \\
\frac{2}{x_{1,0} f_{1}(x_{1,0})} \exp \left[ -\left( \frac{x_{1,0}}{R} \right)^2 Df \right], & \text{(C2 > 1)}.
\end{cases}
\]

Here

\[
C_1 = \frac{2}{x_{1,0} f_{1}(x_{1,0})} \exp \left[ -\left( \frac{x_{1,0}}{R} \right)^2 Df \right],
\]

\[
C_2 = \frac{4}{\pi} \exp \left[ -\left( \frac{n \pi}{Z} \right)^2 Df \right]
\]

and \( t_f \) is the amount of time for the temperature of the center of the cake to become equal to \( T_f \).

This simplification to Eq. (2) is most readily verified by direct calculation. We can solve Eq. (5) explicitly for the baking time, \( t_f \).