REGULARITY PROPERTIES OF FLOWS THROUGH POROUS MEDIA*
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Consider a homogeneous gas flowing through a homogeneous porous medium and suppose that the equation of state has the form

\[ \gamma = \gamma_0 p^\alpha, \]

where \( \gamma \) is the density of the gas at any point, \( p \) the pressure, and \( \gamma_0 \) a constant. If the flow is isothermic then \( \alpha = 1 \), while if it is adiabatic then \( 0 < \alpha < 1 \). Dynamically, the motion is characterized by conservation of mass,

\[ \text{div} (\gamma \mathbf{v}) = -f \frac{\partial \gamma}{\partial t}, \]

and Darcy’s law [1],

\[ \mathbf{v} = -\frac{k}{\mu} \text{grad} p, \]

where \( \mathbf{v} \) is the velocity vector, \( \mu \) the viscosity of the gas, \( f \) the porosity of the medium, and \( k \) the permeability of the medium. Set \( m = 1 + \alpha^{-1} \). If we combine the above equations and normalize so as to reduce the resulting constant to unity, we obtain the equation

\[ \Delta u^m = \frac{\partial u}{\partial t}, \]

where \( u \) is essentially the density of the gas and \( \Delta \) is the Laplace operator. Note that, apart from constants, \( \text{grad} u^{m-1} \) is the velocity vector and \( \text{grad} u^{m-1} \) is the flux vector. Thus, in particular, \( u^{m-1} \) is essentially the pressure which, by Darcy’s law, is also the velocity potential.

For \( m > 1 \), (1) is a nonlinear equation which is parabolic for \( u > 0 \), but which degenerates when \( u = 0 \). The most striking manifestation of the degeneracy of this equation is the finite speed of propagation of disturbances. Thus, if at some instant of time a solution \( u \) of (1) has compact support, then it will continue to have compact support for all later times. In general, the transition from a region where \( u > 0 \) to one where \( u = 0 \) is not smooth and it is therefore necessary to interpret the term “solution of (1)” in some generalized sense. Our object in this paper is to study the regularity properties of a class of generalized solutions of the Cauchy problem for (1). We show that, with respect to the space variables, the velocity potential is Lipschitz continuous, the flux is continuous, and the density is Hölder continuous.

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These statements hold, in particular, across the boundary of the set in which the density is positive. Thus the generalized solutions of (1) which we consider have the properties expected of physical flows.

To avoid unnecessary technical difficulties we restrict our attention to flows in one space variable. Equation (1) is a special case of the equation \[ \{ \phi(x, u) \}_{xx} = u_t \]
 studied by Oleinik, Kalashnikov and Yui-Lin in [4]. (A brief account of their work together with results on other degenerate problems is available in English in Professor Oleinik’s Rome lectures [2].) Let \( S \) denote the strip \( (-\infty, +\infty) \times (0, T] \) in the two-dimensional \( x,t \)-space for some fixed \( T > 0 \) and consider the Cauchy problem

\[
(u^m)_{xx} = u_t \quad \text{for} \quad (x, t) \in S, \quad u(x, 0) = u_0(x) \quad \text{for} \quad -\infty < x < \infty,
\]

where \( u_0 \) is a given bounded, continuous, nonnegative function on the real line. Here we assume only \( m > 1 \). A function \( u = u(x, t) \) is said to be a weak solution of the Cauchy problem (2) in \( S \) if (i) \( u \) is bounded, continuous and nonnegative in \( \bar{S} \), (ii) \( u^m \) possesses a bounded weak (distribution) derivative with respect to \( x \) in \( S \), and (iii) \( u \) satisfies the integral identity

\[
\int_S \int_S \{ \psi_x(u^m)_x - \psi u \} \, dx \, dt = \int_{-\infty}^\infty \psi(x, 0)u_0(x) \, dx
\]

for all \( \psi \in C^1(\bar{S}) \) which vanish for large \( |x| \) and for \( t \) near \( T \). If \( u_0^m \) is Lipschitz continuous, then the corresponding Cauchy problem (2) has a unique weak solution. Moreover, this solution satisfies the equation in the classical sense in a neighborhood of every point \( (x, t) \in S \) at which the solution is positive. These statements are proved in [4], where the solution \( u \) is obtained as the pointwise limit of a decreasing sequence of smooth positive classical solutions of certain boundary value problems related to (2). As part of this construction the authors obtain bounds for \( u \) and Lipschitz constant with respect to \( x \) for \( u^m \) in terms of the corresponding bounds for the data \( u_0 \). Note that since \( m > 1 \) the Lipschitz continuity of \( u^m \) implies that \( u \) is Hölder continuous with exponent \( m^{-1} \). We show that, with respect to \( x \), the velocity potential \( u^{m-1} \) is Lipschitz continuous, the flux \( (u^m)_x \) is continuous and the density \( u \) is Hölder continuous with exponent \( \min \{1, (m - 1)^{-1} \} \). This exponent is best possible as is shown by examples of explicit solutions which will be discussed later. Moreover, for \( m < 2 \), \( u_x \) actually exists and is continuous.

Let \( u \) be a smooth positive classical solution of

\[
(u^m)_{xx} = u_t
\]

in a rectangle \( R = (a, b) \times (0, T] \) and let \( M = \max_R u \). By “smooth” we mean at least \( u \in C^2(R) \cap C^0(\bar{R}) \) and \( u_{xxx} \in C^0(R) \). If \( v \) denotes the velocity potential \( u^{m-1} \), then \( v \) satisfies the nonlinear degenerate parabolic equation

\[
v_t = mvv_{xx} + \frac{m}{m - 1}v_x^2
\]

in \( R \). We seek a bound for \( |v_x| \) which is independent of the lower bound for \( u \).
LEMMA. Let $u$ be a smooth positive classical solution of (3) in $R$ and let $R^* = (a_1, b_1) \times (\tau, T]$ be any proper subrectangle of $R$. Then

$$|v_\alpha(x, t)| \leq \mathcal{C}$$

in $\bar{R}^*$, where $\mathcal{C}$ is a positive constant which depends only on $m, M, a_1 - a, b - b_1$ and $\tau$. If

$$M_1 = \max_{[a,b]} \left| \frac{\partial}{\partial x} u^{m-1}(x, 0) \right| < \infty,$$

then the same conclusion holds for $R^* = (a_1, b_1) \times (0, T]$, where now $\mathcal{C}$ depends on $M_1$ instead of $\tau$.

Proof. For $0 \leq r \leq 1$ let

$$\varphi(r) = \frac{Nr}{3}(4 - r),$$

where $N = M^{m-1}$. Note that $\varphi$ increases from 0 to $N$ as $r$ increases from 0 to 1, and that $\varphi'(r) \geq 2N/3$. Since $0 < \nu \leq N$, the equation $v = \varphi(w)$ defines a function $w = w(x, t)$ with values in $(0, 1]$ which is as smooth as $v$. It follows from (4) that $w$ satisfies the differential equation

$$w_t = m\varphi w_{xx} + m\varphi \frac{\varphi''}{\varphi'} w_x^2 + \frac{m}{m - 1} \varphi' w_x^2$$

in $R$. Moreover, the smoothness of $u$ carries over to $v$ and hence to $w$. Set $p = w_x$. If we differentiate the last equation with respect to $x$ and multiply both sides of the resulting equation by $p$, we obtain

$$\frac{1}{2}(p^2)_t - m\varphi pp_{xx} = \left\{ \frac{m^2}{m - 1}\varphi'' + m\varphi \frac{\varphi''}{\varphi'} \right\} p^4$$

$$+ \left\{ \frac{m(m + 1)}{m - 1}\varphi' + 2m\varphi \frac{\varphi''}{\varphi} \right\} p^2 p_x$$

in $R$.

Let $\zeta = \zeta(x, t)$ be a $C^2(\bar{R})$ function such that $\zeta = 1$ on $\bar{R}^*$, $\zeta = 0$ in a neighborhood of the lower and lateral boundaries of $R$, and $0 \leq \zeta \leq 1$. Set $z = \zeta^2 p^2$. At a point $(x_0, t_0) \in R$ where $z$ attains a maximum we have

$$z_x = 2\zeta^2 p p_x + 2\zeta \zeta_x p^2 = 0$$

and

$$m\varphi z_{xx} - z_t \leq 0.$$

More explicitly, the last inequality takes the form

$$\left( \frac{1}{2}(p^2)_t - m\varphi pp_{xx} \right) \geq m\varphi \zeta^2 p_x^2 + 4m\varphi \zeta \zeta_x p_x + m\varphi \zeta_x^2 p^2 + m\varphi \zeta_{xx} p^2 - \zeta_t p^2.$$

Note that

$$|4m\varphi \zeta \zeta_x p_x| \leq m\varphi \zeta^2 p_x^2 + 4m\varphi \zeta_x^2 p^2.$$
If we use this remark together with (6) and (7) in (5), we obtain the inequality

$$-\frac{m^2}{m-1}\zeta^2 \varphi'' p^4 - \zeta^2 m \varphi \left(\frac{\varphi'}{\varphi}\right) p^4 \leq \{\zeta \xi_{11} - m \varphi \xi_{xx} + 3 m \varphi \zeta_x^2\} p^2$$

$$- \zeta \xi_x \left\{\frac{m(m+1)}{m-1} \varphi' + 2 m \varphi \frac{\varphi''}{\varphi'}\right\} p^3,$$

which holds at \((x_0, t_0)\).

It is easily verified that for \(w \in [0, 1]\) we have \(2N/3 \leq \varphi' \leq 4N/3, \varphi'' = -2N/3, |\varphi''/\varphi'| \leq 1\) and \((\varphi''/\varphi')' \leq -1/4\). Thus, in particular, the coefficient of \(\zeta^2 p^4\) in the first term on the left in (8) is strictly positive, while that of the second term is non-negative. If we drop this latter term, then (8) implies that

$$2\zeta^2 p^4 \leq \mathcal{C}_1 p^2 + \zeta \mathcal{C}_2 |p|^3$$

at \((x_0, t_0)\), where the \(\mathcal{C}_i\) are positive constants which depend only on \(m, N, a_1 - a, b - b_1\) and \(\tau\). Since

$$\mathcal{C}_2 |p|^3 \leq \zeta^2 p^4 + \frac{\mathcal{C}_2^2}{4} p^2,$$

it follows that

$$z(x, t) \leq \max_R z(x, t) \leq \mathcal{C}_1 + \frac{\mathcal{C}_2^2}{4}$$

Therefore

$$\max_{R^*} |w_x| \leq \mathcal{C}_3^{1/2}.$$ 

Finally, \(v_x = \varphi'(w)w_x\) and \(\varphi' \leq 4N/3\) imply that

$$\max_{R^*} |v_x| \leq \frac{4N}{3} \mathcal{C}_3^{1/2}.$$ 

This completes the proof of the first assertion of the lemma. The proof of the second assertion is similar. The main difference is that now we take \(\zeta = \zeta(x) \in C^2_0([a, b])\) with \(\zeta = 1\) on \([a_1, b_1]\) and \(0 \leq \zeta \leq 1\). We omit further details.

The constant \(\mathcal{C}_3\) in the computation given above can be estimated by

$$38 \max \zeta_x^2 + 3 \max |\zeta_x| + \frac{3}{mN} \max |\zeta|.$$ 

Thus the bound for \(|v_x|\) tends to zero with \(N\) at least as fast as \(N^{1/2}\).

The method which we have employed in proving the lemma is a familiar one in the literature on nonlinear parabolic equations (see, for example, [3] and further references given there). In general, it is applicable only to nondegenerate equations; however, we are able to apply it successfully here because of the particular structure of (4).

As we noted above, it is shown in [4] that the weak solution \(u\) of the Cauchy problem (2) is the pointwise limit of a decreasing sequence of positive functions
\{w_n\}. Given any rectangle $R$, the $w_n$ satisfy the hypothesis of the lemma in $R$ provided that $n$ is sufficiently large. Moreover, $w_n \leq 1 + \sup u_0$ for all $n$ at every point of $S$. Thus the bound for the $w_n$ in $R$ is independent of $n$ and the location of $R$. Using this observation, we apply the lemma to obtain the following result.

**Theorem.** Let $u$ be the weak solution of the Cauchy problem (2) in $S$, where it is assumed that $u_0^0$ is Lipschitz continuous.

(i) If $\tau > 0$, then

\begin{equation}
|u^{m-1}(x, t) - u^{m-1}(x', t)| \leq \mathcal{C}_1|x' - x| \tag{9}
\end{equation}

and

\begin{equation}
|u(x, t) - u(x', t)| \leq \mathcal{C}_2|x' - x|^\nu \tag{10}
\end{equation}

hold for all $(x, t), (x', t) \in (-\infty, \infty) \times [\tau, T]$, where $\nu = \min \{1, (m-1)^{-1}\}$ and the $\mathcal{C}_i$ are positive constants which depend only on $m, \tau$ and $\sup u_0$. If $u_0^0$ is Lipschitz continuous, then the same conclusions hold for all $(x, t), (x', t) \in S$, where now the $\mathcal{C}_i$ depend on the Lipschitz constant for $u_0^0$ instead of $\tau$.

(ii) The derivative $\partial u/\partial x$ exists and is continuous as a function of $x$ everywhere in $S$, and, in particular, $\partial u^m(x, t)/\partial x = 0$ if $u(x, t) = 0$.

(iii) If $1 < m < 2$, then $\partial u/\partial x$ exists and is continuous as a function of $x$ everywhere in $S$, and, in particular, $\partial u(x, t)/\partial x = 0$ if $u(x, t) = 0$.

**Proof.** The estimate (9) follows easily from the lemma applied to $w_n^{m-1}$ and the convergence of $w_n^{m-1}$ to $u^{m-1}$. For $m > 2$, (10) is an immediate consequence of (9) and the observation that

\[|u(x, t) - u(x', t)|^m \leq |u^{m-1}(x, t) - u^{m-1}(x', t)|.\]

Now suppose $m \leq 2$. Since

\[\frac{\partial}{\partial x}w_n^{m-1} = (m - 1)w_n^{m-2}\frac{\partial}{\partial x}w_n\]

and $0 < w_n \leq 1 + \sup u_0 = M_1$, the lemma implies that

\[\left|\frac{\partial w_n}{\partial x}\right| \leq \frac{M_1^{2-m}}{m - 1} \mathcal{C} = \mathcal{C}'.\]

Thus

\[|w_n(x, t) - w_n(x', t)| = \left|\int_{x'}^x \frac{\partial}{\partial x}w_n(\xi, t)\, d\xi\right| \leq \mathcal{C}'|x - x'|,
\]

and we obtain (10) by letting $n \to \infty$.

If $u(x, t) > 0$ and $t > 0$, then, according to the results established in [4], $\partial u/\partial x$ exists and is continuous in a neighborhood of $(x, t)$ and the same is true for $\partial u^m/\partial x$. Therefore, it suffices to prove (ii) only at those points $(x_0, t_0)$ for which $u(x_0, t_0) = 0$. Let $I_\delta(x_0)$ denote the interval $|x - x_0| < \delta$. In view of (10), $0 \leq u(x, t_0) \leq \mathcal{C}_2 \delta^\nu$ for all $x \in I_\delta(x_0)$. The sequence of continuous functions $\{w_n\}$ decreases to the continuous function $u$. Hence, by Dini's theorem, $w_n \to u$ uniformly in any
compact subset of $S$. Thus, in particular, we have $0 < w_n(x, t_0) \leq 2 \varepsilon_2 \delta^v$ for all sufficiently large $n$ and all $x \in I_0(x_0)$. Now consider

$$w_n^m(x_2, t_0) - w_n^m(x_1, t_0) = \frac{m}{m - 1} \int_{x_1}^{x_2} w_n(x, t_0) \frac{\partial}{\partial x} w_n^{m-1}(x, t_0) \, dx.$$

If $x_1, x_2 \in I_0(x_0)$, then, by the lemma,

$$|w_n^m(x_2, t_0) - w_n^m(x_1, t_0)| \leq \varepsilon_3 \delta |x_2 - x_1|,$$

and, letting $n \to \infty$, we have

$$|u^m(x_2, t_0) - u^m(x_1, t_0)| \leq \varepsilon_3 \delta |x_2 - x_1|.$$

If we take $x_1 = x_0$, then it follows from (11) that $\partial u^m(x_0, t_0)/\partial x$ exists and equals zero. Thus $\partial u^m/\partial x$ exists everywhere in $S$ and is zero at all points where $u = 0$. Now let $x_1 \in I_0(x_0)$ be such that $u(x_1, t_0) > 0$. Since $\partial u^m(x_1, t_0)/\partial x$ exists, (11) implies that

$$|\frac{\partial}{\partial x} u^m(x_1, t_0)| \leq \varepsilon_3 \delta^v.$$

On the other hand, if $x_1 \in I_0(x_0)$ and $u(x_1, t_0) = 0$, then $\partial u^m(x_1, t_0)/\partial x = 0$. Thus (12) holds for all $x_1 \in I_0(x_0)$ and $\partial u^m/\partial x$ is continuous with respect to $x$ at $(x_0, t_0)$.

To prove (iii) it again suffices to consider only those points $(x_0, t_0)$ for which $u(x_0, t_0) = 0$. By the lemma we have

$$|\frac{\partial}{\partial x} w_n^{m-1}(x, t_0)| = (m - 1)w_n^{m-2}(x, t_0) \left| \frac{\partial}{\partial x} w_n(x, t_0) \right| \leq \varepsilon.$$

As before, if $x \in I_0(x_0)$ and $n$ is sufficiently large, then $0 < w_n(x, t_0) \leq 2 \varepsilon_2 \delta$. Here $v = 1$ since, by hypothesis, $m < 2$. It follows that

$$|\frac{\partial}{\partial x} w_n(x, t_0)| \leq \varepsilon_4 \delta^{2-m}$$

for $x \in I_0(x_0)$. Therefore

$$|u(x_2, t_0) - u(x_1, t_0)| = \lim_{n \to \infty} |w_n(x_2, t_0) - w_n(x_1, t_0)| \leq \varepsilon_4 \delta^{2-m} |x_2 - x_1|$$

for all $x_1, x_2 \in I_0(x_0)$. The remainder of the proof is exactly the same as the corresponding part of the proof (ii) and we omit further details.

Note that we have actually shown that $\partial u^m/\partial x$ and $\partial u/\partial x$ for $m < 2$ have Hölder moduli of continuity at points where $u = 0$. The same is true at points where $u > 0$, but we are, at present, unable to give estimates. For this purpose it would seem to be necessary to obtain a characterization of the boundary of the set where $u > 0$ and to investigate the smoothness of $u$ in the closure of this set. There also remains the question of the smoothness of $u$ with respect to $t$.

There are several examples of explicit solutions of (3) in the literature (see [2] and [4] for further references). The following example is due to Pattle [5].
Define

\[ \lambda(t) = \left( \frac{2m(m+1)}{m-1} \frac{(t+1)}{} \right)^{1/(m+1)} \]

for \( t \geq 0 \). Then

\[ u(x, t) = \begin{cases} \frac{1}{\lambda(t)} \left[ 1 - \left( \frac{x}{\lambda(t)} \right)^2 \right]^{1/(m-1)} & \text{for } |x| \leq \lambda(t), \quad t \geq 0, \\ 0 & \text{for } |x| > \lambda(t), \quad t \geq 0, \end{cases} \]

is a weak solution of the Cauchy problem (2) with initial data

\[ u_0(x) = \begin{cases} \frac{1}{\lambda(0)} \left[ 1 - \left( \frac{x}{\lambda(0)} \right)^2 \right]^{1/(m-1)} & \text{for } |x| \leq \lambda(0), \\ 0 & \text{for } |x| > \lambda(0). \end{cases} \]

It is clear that \( \partial u^{m-1}/\partial x \) is discontinuous at \( x = \pm \lambda(t) \). Thus the Lipschitz continuity of \( u^{m-1} \) is the best possible global result. Moreover, since

\[ u(x, t) - u(\lambda(t), t) = \left. \{ \lambda(t) \}^{(m+1)/(1-m)} \{ \lambda(t) + x \} \{ \lambda(t) - x \} \right]^{1/(m-1)}, \]

the Hölder exponent \( \min \{ 1, (m - 1)^{-1} \} \) in (10) cannot be increased.

Added in proof. S. N. Kruzhkov informs us that he has proved, for a certain class of nonlinear parabolic equations, that Hölder continuity in \( x \) implies Hölder continuity in \( t \). Using this general result together with the results established here, he has been able to show that \( u \) is also Hölder continuous in \( t \) with exponent \( \nu/(2 + \nu) \). Kruzhkov’s work will appear in a paper entitled “Results on the character of continuity of solutions of parabolic equations and some of their applications” to be published in the journal Matematicheskii Sbornik.

REFERENCES


