Weakly Connected Oscillatory Networks for Dynamic Pattern Recognition

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Abstract—Recent studies on the thalamo-cortical system have shown that weakly connected oscillatory networks (WCNs) exhibit associative properties and can be exploited for dynamic pattern recognition. In this manuscript we focus on WCNs, composed of oscillators that admit of a Lur'e like description and are organized in such a way that they communicate one another, through a common medium. The main dynamic features are investigated by exploiting the phase deviation equation (i.e. the equation that describes the phase deviation due to the weak coupling). Furthermore, by using a simple learning algorithm, the phase-deviation equation is designed in such a way that given sets of patterns can be stored and recalled. In particular, two models of WCNs associative and dynamic memories are provided.

I. INTRODUCTION

Nonlinear oscillatory networks have been widely used in Biology, Physics and Engineering for modeling complex space-time phenomena [1]. The most significant and worth studying property of oscillatory systems is synchronization, either when they are coupled with other oscillators or when they are subject to an external driving signal.

In this work we focus on weakly connected oscillatory networks that represent bio-inspired architectures for information and image processing. Recent studies in neuroscience have shown that some significant features of the visual systems, like the binding problem [2], can be investigated, by exploiting nonlinear dynamic network models [3]. Some studies on the thalamo-cortical system have suggested new architectures for neurocomputers, that consist of coupled arrays of oscillators, with a periodic and/or complex dynamic behavior (including the possibility of chaos) [4], [5]. In particular, it has been shown that nonlinear oscillatory networks can behave as Hopfield neural networks, whose attractors are limit cycles instead of equilibrium points [4], [5].

The mathematical model of a weakly connected oscillatory network (WCN) consists of a large system of coupled nonlinear ordinary differential equations (ODEs), that may exhibit a rich spatio-temporal dynamics, including several attractors and bifurcation phenomena [6]. For this reason WCN dynamics has been mainly investigated through time-domain numerical simulation. Recently some spectral techniques have been applied to space-invariant networks, in order to characterize some space-time phenomena (see [6] and in particular [7], [8]). However the proposed methods are not suitable for characterizing the global dynamic behavior of complex networks, that exhibit a large number of attractors.

The global dynamic behavior of WCNs can be investigated through the phase deviation equation [5], i.e. the equation that describes the evolution of the phase deviations, due to the weak coupling. We have employed this method for investigating one-dimensional weakly connected networks, composed by third order oscillators (Chua’s circuits) [9].

In this manuscript we consider WCNs composed by oscillators that admit of a Lur’e like description and organized in such a way that each oscillator communicates with the others through a central system (called master cell).

As shown in [4] and [10], such networks can be employed as oscillatory associative memories because there is a one to one correspondence between the equilibrium points of the phase deviation equation and the limit cycles of the WCN.

Firstly, we derive an accurate analytic expression of the phase deviation equation, by extending the technique presented in [9]. Then we show that the equilibrium points of the phase deviation equation can be designed through a simple learning rule in order to retrieve a given set of stored pattern. In particular, we presents two WCN models such that: (a) the outputs of the oscillators are only in-phase or anti-phase; (b) the outputs of the oscillators are not in-phase or anti-phase.

II. WEAKLY CONNECTED NETWORKS

We consider weakly connected networks (WCNs) [5], having the star topology [10], that originates from the bio-inspired architecture proposed in [4]. All the cells are connected to a central complex cell $O_0$ (called master cell) in the shape of a star and communicate each other only through the central system.

Let us assume that each cell $O_i$ is a dynamical system of order $m$ described by the following system of nonlinear ordinary differential equations (ODEs) $(1 \leq i \leq n)$:

$$\dot{X}_i = F_i(X_i), \quad X = [X_1^T, \ldots, X_n^T]^T \quad (1)$$

where $X_i \in \mathbb{R}^m$ represents the state vector of each cell, $F_i : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $T$ denotes transposition.

The cells $O_i$ $(1 \leq i \leq n)$ interact only through the master cell that supplies the signal $G_i(X_0, X)$ to each cell, where $G_i : \mathbb{R}^{m(n+1)} \rightarrow \mathbb{R}^m$ and $X_0^T$ is the state vector of the master cell whose dynamics is described by $X_0 = F_0(X_0)$ with $X_0 \in \mathbb{R}^m$.

Star topology WCNs, composed by $n$ cells and one master cell, are then described by $\dot{X}_i = F_i(X_i) + \varepsilon G_i(X_0, X)$ where $0 \leq i \leq n$ and $\varepsilon$ represents a small parameter that guarantees a weak connection among the cells and $G_0(X_0, X) = 0$.
By assuming that each uncoupled cell and the master cell admit of a Lur'e representation [11], the resulting WCN can be described by the following simplified system of Lur'e like equations (see [9] for more details):

\[ L^0(D)x^0(t) = f_0[x^0(t)] \]

\[ L^1(D)x^1(t) = f[x^1(t)] + \varepsilon g_i(x^0(t), x^1(t)), \quad (1 \leq i \leq n) \quad (3) \]

where \( x^i \in \mathbb{R} \) is a scalar component of \( X_i \), \( X^i \in \mathbb{R}^{m-1} \) represents the collection of the other components of \( X_i \).

It turns out that only one component of \( G_i \) \((1 \leq i \leq n)\) is different from zero, i.e.,

\[ G_i(X_0, X) = (g(x^0_i, x^1_i), 0, \ldots, 0)^T \]

whereas all the components of \( G_0 \) are zero:

\[ G_0(X_0, X) = (0, 0, \ldots, 0)^T . \]

A. Joint application of the Malkin's Theorem and the describing function technique

In order to compute the phase deviation equation by applying the Malkin's Theorem to weakly connected networks, the authors proposed in [9] a method summarized by the following three steps.

1) Describing function approximation of \( \gamma_i(t) \): Each variable \( x^i(t) \) \((0 \leq i \leq n)\) of the Lur'e model (2)-(3) is approximated through the describing function technique (see [9] for details): \( x^i(t) \approx \tilde{x}^i(t) = A_i + B_i \sin(\omega_i t) \), where \( A_i \) denotes the bias, \( B_i \) the amplitude of the first harmonic, and \( \omega_i \) the angular frequency.

2) Describing function approximation of \( Q_i(t) \): According to Malkin's Theorem, \( Q_i = \left[ Q^0_i, Q^1_i \right]^T \) is the unique \( T \)-periodic solution to the linear time-variant system (7), satisfying the normalization condition (8). As shown in [9], the following harmonic approximation \( \tilde{Q}^i(t) \) of \( Q^i(t) \) can be written as: \( \tilde{Q}^i(t) = \delta_i(\omega_i) \cos(\omega_i t) \), where \( \delta_i(\omega_i) \) can be analytically derived (see [9] for more details).

3) Phase deviation equations: We show that an explicit and very accurate expression of the phase deviation equation can be derived, by substituting in (6) the describing function approximation of \( \gamma_i(t) \) and \( Q_i(t) \), provided by the previous steps.

By using (5) it is easily derived that \( \phi^0_0 = 0 \). Thus, assumption (5) implies that the oscillators \( O_i \) do not influence the dynamics of the master cell \( O_0 \), i.e. \( \phi_0 \) in absence of couplings do not change for weak couplings. On the other hand, by remembering that (4) has only one component of \( G(X_0, X) \) different from zero, we obtain (1 \( \leq i \leq n)\):

\[ \phi_i = H_i(\phi - \phi_0, 0) = \frac{1}{T} \int_0^T \tilde{Q}^i(t)G_i \left[ \left( \frac{t + \frac{\phi}{\omega_i}}{\omega_i} \right) \right] dt \]

\[ \gamma(t + \frac{\phi}{\omega_i}) = \left[ y^0_i \left( t + \frac{\phi - \phi_0}{\omega_i} \right), \gamma_1^i \left( t + \frac{\phi - \phi_0}{\omega_i} \right), \ldots, \gamma_{n-1}^i \left( t + \frac{\phi - \phi_0}{\omega_i} \right) \right]^T \]

where \( T \) is the minimum common multiple of \( T_0, T_1, \ldots, T_n \).

In the above expression (6) \( Q_i(t) \in \mathbb{R}^{m} \) is the unique non-trivial \( T \)-periodic solution to the linear time-variant system:

\[ \dot{Q}_i(t) = -DF_i(\gamma_i(t))^T \dot{Q}_i(t) \]

\[ \dot{Q}_0^0(0)F_0(\gamma_0(0)) = 1 \]

(7)

(8)
It is worth observing that the functions $g_i(\cdot)$, defining how cells $O_i$ ($1 \leq i \leq n$) are linked, specify the properties of (11) through the coefficients $\lambda^i_k(\eta)$ and $\lambda^j_k(\eta)$.

We are mainly interested to the steady-state solutions of (11), in which the oscillators are synchronized with appropriate phase patterns defined by the stationary solutions of (11).

If we denote with $\eta^*$ a phase configuration pattern such that $\eta^*_j = 0$, that is $\lambda^i_k(\eta^*) \cos(\eta^*_j) = 2 \lambda^j_k(\eta^*) \sin(\eta^*_j)$, the stability properties of $\eta^*$ can be investigated by analyzing the eigenvalues of the Jacobian of (11) evaluated in $\eta^*$.

In the next section we will present two WCN models in which the functions $g_i(\cdot)$ are designed in such a way that the network operates as an associative or dynamic memory.

III. Oscillatory associative and dynamic memories

As shown in the previous section, equation (11) reduces to a rather complex network of oscillators to a simpler model, that may be analytically dealt with. This gives the possibility of developing new applications, that exploit the rich dynamic behavior of nonlinear dynamic arrays, including dynamic pattern recognitions and associative memories [4].

A. Associative memory

Let us assume that the master cell provides a linear interconnections among cells $O_i$, but that it does not influence their dynamics, i.e. that $g_i(\cdot)$ does not depend on its first argument:

$$g_i(x^i_0(t), x^i_1(t), \ldots, x^i_n(t)) = \sum_{j=1}^{n} C_{ij} x^j_0(t)$$

(12)

It is readily obtained that the coefficients $\lambda^i_k(\eta)$ and $\lambda^j_k(\eta)$ are different from zero if and only if all oscillators have the same angular frequency ($\omega_i = \omega_j$ for all $i \neq j$), that is:

$$\lambda^i_k(\eta) = \sum_{j=1}^{n} C_{ij} \delta_i^j B_j \sin \eta_j, \quad \lambda^j_k(\eta) = \sum_{j=1}^{n} C_{ij} \delta_i^j B_j \cos \eta_j$$

By substituting the expressions above in (11), we get the following simple Kuramoto-like model ($s_{ij} = \frac{C_{ij} \delta_i^j B_j}{2}$):

$$\eta^*_j = \sum_{i=1}^{n} s_{ij} \sin \eta_j, \quad \eta^*_i = \sum_{i=1}^{n} s_{ij} \cos \eta_j.$$  

(13)

Each limit cycle (either stable or unstable) of the WCN corresponds to an equilibrium point of the phase deviation equation (13) (see [5]). The total number of periodic limit cycles and their stability properties can be revealed by exploiting the analytical Kuramoto-like form of (13) (see [9]). It is readily derived that phase configurations such that $(\eta_j - \eta_i) \in [0, \pi]$ are equilibrium points of (13), that is the oscillators can be in-phase or anti-phase.

In order to show the oscillatory associative properties of the WCN associated to the phase deviation (13), let us assume that a given set of $p$ phase patterns have to be memorized:

$$\psi^k = [\psi^k_1, \psi^k_2, \ldots, \psi^k_n], \quad k = 1, 2, \ldots, p$$

(14)

where $\psi^k_j = \psi^k_1$ and $\psi^k_j = -\psi^k_i$ if the $i$th and $j$th oscillators are in-phase or anti-phase, respectively.

Binary phase patterns, for a WCN composed of 25 identical cells (plus the master cell) and cast into a regular grid with 5 rows and 5 columns, are shown in Figure 1.

Among many possible learning algorithms the simplest one that can be considered is the Hebbian learning rule, that is the coefficients $s_{ij}$ of (13) are designed as:

$$s_{ij} = \frac{1}{n} \sum_{k=1}^{n} \psi^k_i \psi^k_j.$$  

(15)

Computer simulations (not reported here for the lack of space) show that the system given by (13) and (15), starting from a given initial phase configuration, converges towards a phase equilibrium point having phase relations $0$ or $\pi$ in accordance with the phase relations defined by a stored or a spurious pattern. Thus, the network can store and retrieve oscillatory patterns, consisting of periodic limit cycle, such that the oscillators result only in-phase or anti-phase.

Hence, system (13) with the simple Hebbian learning rule can be used to realize and design Hopfield-Grossberg-like associative memories [4]. Numerical simulations have also confirmed that the storage capacity of the oscillatory associative memory is equivalent to that of the Hopfield memory.

B. Dynamic memory

Let us consider a star WCN in which the master cell provides to each cell the following signal (see [10]):

$$g_i(x^i_0(t), x^i_1(t), \ldots, x^i_n(t)) = \left(\sum_{j=1}^{n} C_{ij} \text{sgn}(x^j_0(t)) \right) |x^i_0(t)| - x^i_0(t)$$

(16)

where $\text{sgn}(\cdot)$ denotes the sign function.

As shown in [10], star networks, whose cells interact according to functions (16), act as dynamic memories, i.e. the output patterns, given in the form of synchronized chaotic states, can travel around stored and spurious patterns.

We will show that a similar behaviour can be obtained by considering an oscillatory star WCN with the connections among the cells defined by functions (16).

Let us assume that the star WCN is composed by identical cells and each oscillatory cell has at least an hyperbolic symmetric limit cycle, that is each uncoupled dynamical system has at least a periodic orbit surrounding the origin. It is worth noting that the last assumption is easily satisfied by
Chua’s or Van der Pol’s oscillators for wide ranges of circuitual parameters.

By focusing on the set of initial conditions such that each uncoupled cell oscillates on the symmetric limit cycle, the phase deviation equations (11) can be written as (1 ≤ i ≤ n):

$$\eta_i' = \delta B \left[ \sum_{j \neq i} C_{ij} \xi_{ij}(\eta_j) \cos(\omega_j) - \sum_{j \neq i} C_{ij} \xi_{ij}(\eta_j) \sin(\omega_j) + \frac{\sin(\omega_j)}{2} \right]$$  (17)

where $\omega_i = \omega$, $\delta_i = \delta$ and $B_i = B$ for all 0 ≤ i ≤ n because all the cells are identical, while $\xi_{ij}(\eta)$ and $\xi_{ij}(\eta)$ are defined as:

$$\xi_{ij}(\eta) = \frac{1}{T} \int_0^T \cos(\omega \tau') \sin(\omega \tau') \text{sgn} \left[ \sin(\omega \tau' + \eta_j) \right] d\tau'$$  (18)

$$\xi_{ij}(\eta) = \frac{1}{T} \int_0^T \sin(\omega \tau') \sin(\omega \tau') \text{sgn} \left[ \sin(\omega \tau' + \eta_j) \right] d\tau'$$  (19)

The coefficients above are given as the integral of functions that are periodic of period $T$ in the variable $\tau'$. It follows that, $\xi_{ij}(\eta)$ and $\xi_{ij}(\eta)$ do not depend on the variable $\eta_j$, that is they are only function of $\eta_j$. By integrating over the interval [0, $T$], (18) and (19) can be analytically computed:

$$\xi_{ij}(\eta) = \frac{1}{2 \pi} \left[ 1 - \cos(2\eta_j) \right] p(\eta_j)$$  (20)

$$\xi_{ij}(\eta) = \frac{1}{2 \pi} \left[ \pi - m_{2\eta}(2\eta_j) + \sin(2\eta_j) \right] p(\eta_j)$$  (21)

where $m_{2\eta}(2\eta_j) = 2\eta_j - 2\pi \left[ \frac{j}{2} \right]$ and $p(\eta_j)$ is a square wave with period 2$\pi$ defined as: $p(\eta_j) = +1$ if $\eta_j \in [0, \pi)$ and $p(\eta_j) = -1$ if $\eta_j \in [\pi, 2\pi)$.

Finally, the substitution of (20) and (21) in (17) allows us to obtain the following phase deviation equations for the star WCN with connections defined by (16) ($s_{ij} = C_{ij} \frac{\delta B}{2\pi}$):

$$\eta_i = \sum_{j=1}^{n} s_{ij} \xi_{ij}(\eta_j) \cos \eta_j - \sum_{j=1}^{n} s_{ij} \xi_{ij}(\eta_j) \sin \eta_j + \frac{\delta B}{2} \sin \eta_j.$$  (22)

It is derived that the phase patterns composed by sequence of 0 and/or $\pi$ are equilibrium points of the phase deviation equation (22). Furthermore, by designing the coefficients $s_{ij}$ according to (15) we can also obtain stable equilibrium points whose phase components are different from 0 or $\pi$.

Figure 2 points out that the phase deviation equation (22) may converge towards an equilibrium point with phase components different from 0 or $\pi$. It follows that the WCN, related to (22), has a corresponding limit cycle whose components have phase shifts defined by the phase equilibrium point.

By noting that the output of each oscillator is given through the sign function, we derive that the WCN oscillatory output travels around a finite sequence of binary stored and spurious patterns. Thus, an oscillatory star WCN with the connections among the cells defined by the functions (16) can operate as a dynamic memory (see [10]).

IV. CONCLUSIONS

Weakly connected oscillatory networks (WCNs) are bio-inspired models having associative properties that can be exploited for dynamic pattern recognition. We have considered WCNs, composed by nonlinear oscillators that admit of a Lur’e model, having the star topology. We have shown that an accurate analytical expression of the phase deviation equation can be derived via the joint application of the describing function technique and of Malkin’s Theorem. The proposed technique can be effectively exploited for designing associative and dynamic memories. In particular, by using the simple Hebbian learning rule, two different WCNs have been proposed, that behave as associative and dynamic memories.

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